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Welcome!

Beyond Invert and Multiply: Making Sense of Fraction Computation

Beyond Invert and Multiply: Making Sense of Fraction Computation

Julie McNamara & Patty Clark

NCTM Philadelphia, PA

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8th Grade, NAEP 1996

Estimate the answer to $\frac{12}{13} + \frac{7}{8}$

Answer choices were:

A.	1	7%
B.	2	24%
C.	19	28%
D.	21	27%
E.	DK/No answer	14%

Mathematical Reasoning Inventory 2011/2012

Is $\frac{11}{12} + \frac{1}{5}$ less than one or greater than one?

30% of students (grades 5 and 6) answered incorrectly.

"The difficulty with fractions (including decimals and percents) is pervasive and is a major obstacle to further progress in mathematics. . ."

—*Report of the National Math Panel*, March 2008

4th Grade & 8th Grade NAEP, 2003

Jim has $3/4$ of a yard of string which he wishes to divide into pieces, each $1/8$ of a yard long. How many pieces will he have?

	4th	8th		4th	8th
A. 3	19%	15%	C. 6	27%	54%
B. 4	30%	21%	D. 8	22%	8%

8th Grade, NAEP 2007

Add the numbers 7/10, 7/100, and 7/1000.

Write this sum as a decimal.

Correct:	46%
Incorrect:	51%
Omitted:	3%

Why are fractions so hard?

- Fraction notation – numbers must be considered in new ways
- Practices that simplify and/or mask the meaning of fractions
- Many meanings of fractions
- Overreliance on whole number knowledge

Without computing the exact answer, decide which of these expressions would produce the answer with the least value and the greatest value.

A. Addition: $3/4 + 5$

B. Subtraction: $3/4 - 5$

C. Multiplication: $3/4 \times 5$

D. Division: $3/4 \div 5$

Least: 61% chose subtraction; 30% chose division
Greatest: 81% chose multiplication

N= 132
Grades 4, 5, & 6

Katherine, Grade 5

1. Without computing the exact answer, decide which of these expressions would produce the answer with the least value and the greatest value.

A. Addition: $\frac{3}{4} + \frac{5}{8}$

B. Subtraction: $\frac{3}{4} - \frac{5}{8}$

C. Multiplication: $\frac{3}{4} \times \frac{5}{8}$

D. Division: $\frac{3}{4} \div \frac{5}{8}$

Least Value Division

Greatest Value Multiplication

Explain your thinking below:

Division is the least value because if you divide you will get a small value. Multiplication will get a great value because if you multiplie you get a great value.

Andres, Grade 4

1. Without computing the exact answer, decide which of these expressions would produce the answer with the least value and the greatest value.

A. Addition: $\frac{3}{4} + \frac{5}{8} = \frac{10}{4}$

B. Subtraction: $\frac{3}{4} - \frac{5}{8} = \frac{1}{4}$

C. Multiplication: $\frac{3}{4} \times \frac{5}{8} = \frac{17}{8}$

D. Division: $\frac{3}{4} \div \frac{5}{8} = \frac{1}{8}$

Least Value $\frac{17}{8}$

Greatest Value $\frac{1}{4}$

Explain your thinking below:

Francisco, Grade 6

1. Without computing the exact answer, decide which of these expressions would produce the answer with the least value and the greatest value.

A. Addition: $\frac{3}{4} + \frac{5}{8} = 1\frac{3}{8}$

B. Subtraction: $\frac{3}{4} - \frac{5}{8} = 1\frac{3}{8}$

C. Multiplication: $\frac{3}{4} \times \frac{5}{8} = \frac{15}{24}$

D. Division: $\frac{3}{4} \div \frac{5}{8} =$

Least Value $1\frac{3}{8}$

Greatest Value $\frac{15}{24}$

Explain your thinking below:

"Students make mistakes not because they are ***not*** thinking, but because they ***are*** thinking."

-Constance Kamii

What is Fraction Sense?

Fraction sense is tied to common sense: Students with fraction sense can reason about fractions and don't apply rules and procedures blindly - nor do they give nonsensical answers to problems involving fractions.

Standards for Mathematical Content (CCSS)
Number and Operations - Fractions

Grade 3:

Develop understanding of fractions as numbers.

"It may be surprising that, for most students, to think of a rational number as a number – as an individual entity or a single point on a number line – is a novel idea."

Adding It Up: Helping Children Learn Mathematics,
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Standards for Mathematical Content (CCSS)
Number and Operations - Fractions

Grade 4:

Extend understanding of fraction equivalence and ordering.

Build fractions from unit fractions by applying and extending previous understandings of operations on whole numbers.

Understand decimal notation for fractions, and compare decimal fractions.

Standards for Mathematical Content (CCSS)
Number and Operations - Fractions

Grade 5:

Use equivalent fractions as a strategy to add and subtract fractions.

Apply and extend previous understandings of multiplication and division to multiply and divide fractions.

Standards for Mathematical Content (CCSS)
The Number System

Grade 6:

Apply and extend previous understandings of multiplication and division to divide fractions by fractions.

Apply and extend previous understandings of numbers to the system of rational numbers.

Standards for Mathematical Content (CCSS)
The Number System

Grade 7:

Apply and extend previous understandings of operations with fractions to add, subtract, multiply, and divide rational numbers.

Grade 8:

Know that there are numbers that are not rational, and approximate them by rational numbers.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Standards for Mathematical Content (CCSS)
Number and Operations – Fractions

Grade 3: Explain **equivalence** of fraction in special cases, and compare fractions by reasoning about their size.

Grade 4: Extend understanding of fraction **equivalence** and ordering.

Grade 5: Use **equivalent** fractions as a strategy to add and subtract fractions.

Grades 6-8: Apply and extend previous understandings…

Fraction Sense and Operations with Fractions

Tell Me All You Can

$$3/4 + 5/8$$

Tell Me All You Can Prompts

- The answer is going to be around/about _____ because_____.
- The answer is going to be close to _____ because_____.
- The answer is going to be between _____ and _____because_____.
- The answer is going to be greater than _____ because_____.
- The answer is going to be less than _____ because_____.

Tell Me All You Can

$$3/4 \quad - \quad 5/8$$

Tell Me All You Can

$$3/4 \times 5/8$$

Tell Me All You Can

$$3/4 \div 5/8$$

How does *Tell Me All You Can* support the development of Fraction Sense?

Find the sum

$$\frac{2}{3} + \frac{1}{4} = \frac{3}{7}$$

Find the sum

$$\frac{2}{3} + \frac{1}{4} = \frac{3}{7}$$

Is the answer greater or less than $\frac{1}{2}$?

Is the answer greater or less than 1?

What else do I know about the answer?

Adding and Subtracting on the Number Line

Number Line Principles

(adapted from the Learning Mathematics through Representations Project at UC Berkeley)

- Numbers increase in value from left to right. Numbers decrease in value from right to left.
- The value of a number is based on its distance from zero.
- The “unit interval” is the distance from 0 to 1 (or any equivalent distance on the line).
- As long as any 2 numbers are shown on the line the unit interval can be determined.

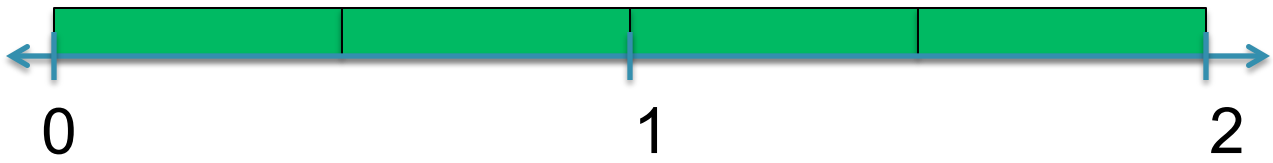
Number Line Principles

(adapted from the Learning Mathematics through Representations Project at UC Berkeley)

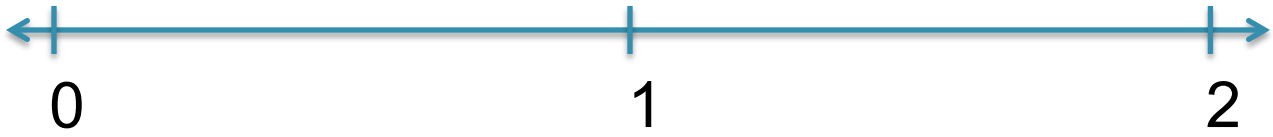
- The unit interval can be divided further into sub-units.
- The denominator of a fraction tells how many sub-units the unit interval has been divided into.
- Every number has a place on the number line, but not every number needs to be shown.

Making a Number Line

- Take out 4 dark green rods.
- Use 2 dark green rods to create your unit interval. → One dark green rod equals ½ of the unit interval.

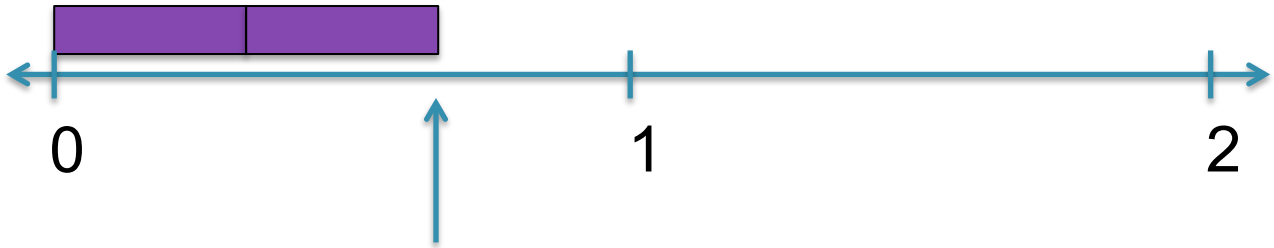


2-Unit Number Line



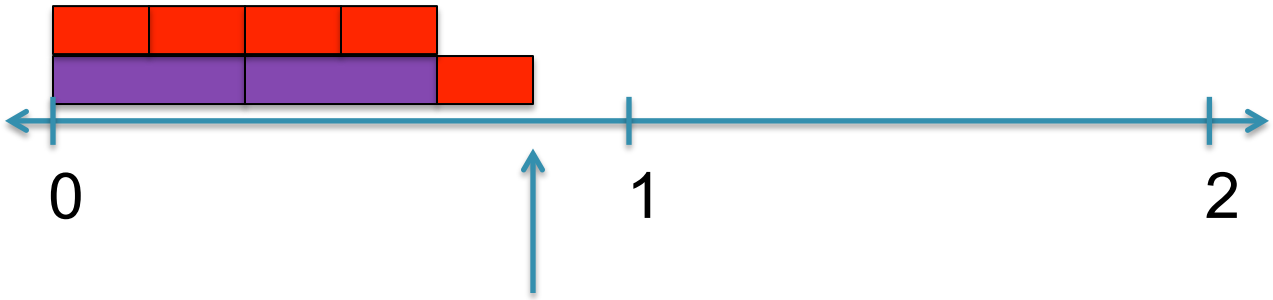
Find the rod that equals $1/3$ of the unit interval

- Use the rods to solve $1/3 + 1/3$



Find the rod that equals $1/6$ of the unit interval

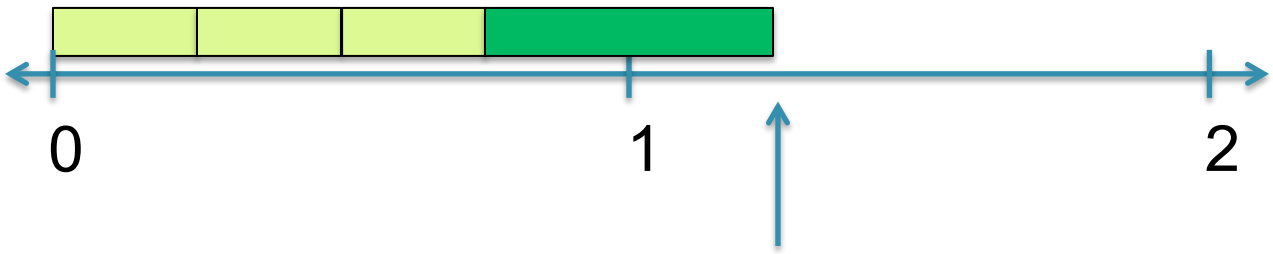
- Use the rods to solve $2/3 + 1/6$



2 /6

Find the rod that equals $1/4$ of the unit interval

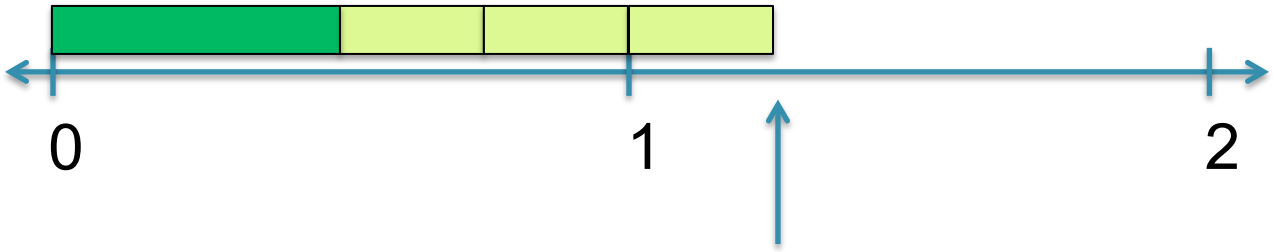
- Use the rods to solve $3/4 + 1/2$



Find the rod that equals $1/4$ of the unit interval

- What happens if I change the problem to :

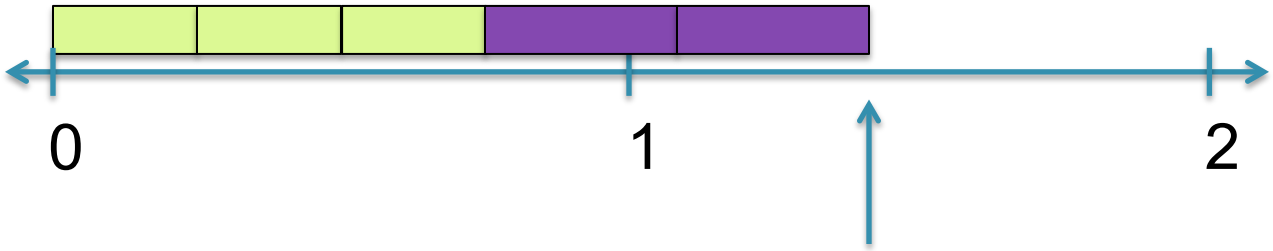
$$1/2 + 3/4$$



What property did I use?

Use the rods that equal 1/4 and 1/3 of the unit interval

- Solve 3/4 + 2/3



use the rods to solve the following problems:

- 3/4 + 5/6
- 1/3 + 1/2
- 2, _ + 5/12
- 3, = + 1/6
- 1/4 + 2/3
- 11, _ 2 + 1/2

Moving beyond the rods…

- How can this experience help students think about adding and subtracting without using the rods?
- What important ideas are students drawing upon?
- What important ideas are students developing?

Standards for Mathematical Content (CCSS)
Number and Operations - Fractions

Grade 5:

Use equivalent fractions as a strategy to add and subtract fractions.

Find the sum

4/6 6/9 12/18 8/12 10/15 2/8 4/16 3/12

$$\frac{2}{3} + \frac{1}{4} = \frac{11}{12}$$

1. Start with finding equivalent fractions.

2. Then find common denominators..

3. Add, and check to see if my answer makes sense.

Finally.....

$$a \times 1 = a$$

$$b/b = 1$$

$$a \times b/b = a$$

Multiplicative identity property of 1

A student does the following multiplication problem:

$$\frac{5}{6} \times \frac{2}{2} = \frac{10}{12}$$

Look at the statement below:

$\frac{10}{12}$ is twice as large as $\frac{5}{6}$.

Decide whether you agree or disagree with the statement.

Agree Disagree

A student does the following multiplication problem:

$$\frac{5}{6} \times \frac{2}{2} = \frac{10}{12}$$

Look at the statement below:

$\frac{10}{12}$ is twice as large as $\frac{5}{6}$.

60% of 4th graders and 51% of 6th graders agreed with the statement.

A student does the following division problem:

$$\frac{6}{10} \div \frac{2}{2} = \frac{3}{5}$$

Look at the statement below:

$\frac{3}{5}$ is half the size of $\frac{6}{10}$.

Decide whether you agree or disagree with the statement.

Agree Disagree

A student does the following division problem:

$$\frac{6}{10} \div \frac{2}{2} = \frac{3}{5}$$

Look at the statement below:

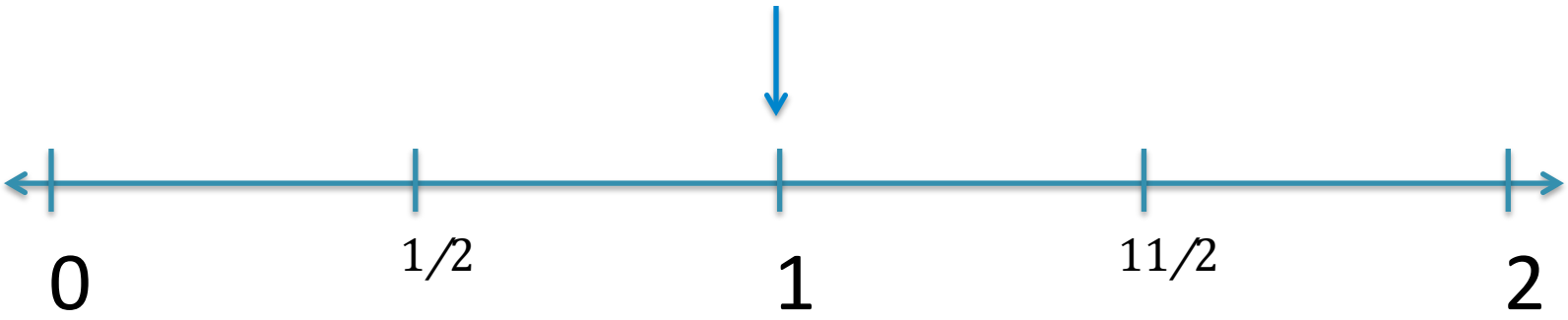
$\frac{3}{5}$ is half the size of $\frac{6}{10}$.

73% of 4th graders and 57% of 6th graders agreed with the statement.

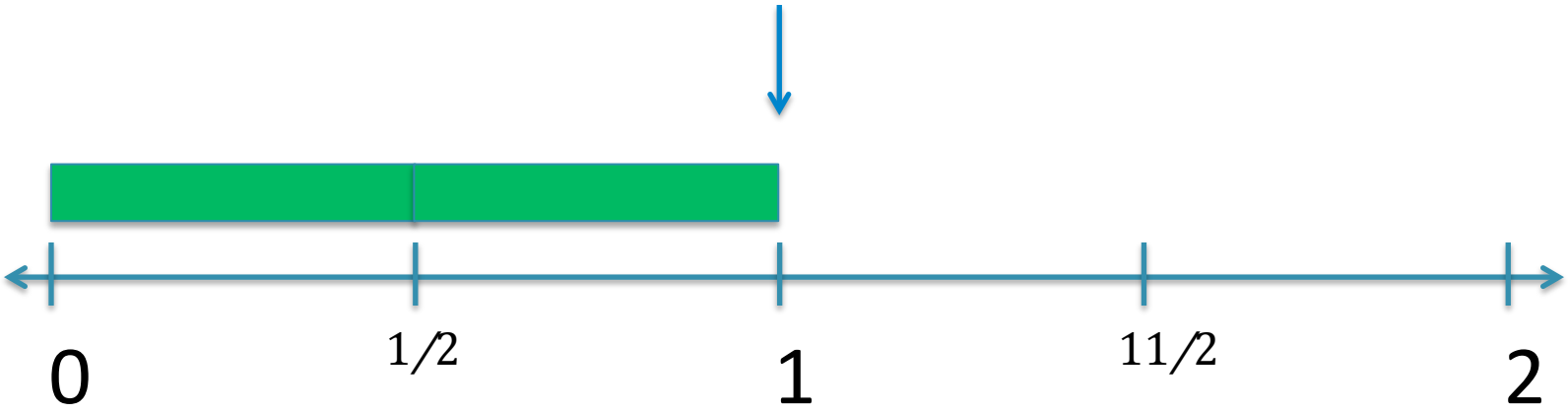
Fraction Sense and Multiplication with the Number Line

$2 \times 1/2$

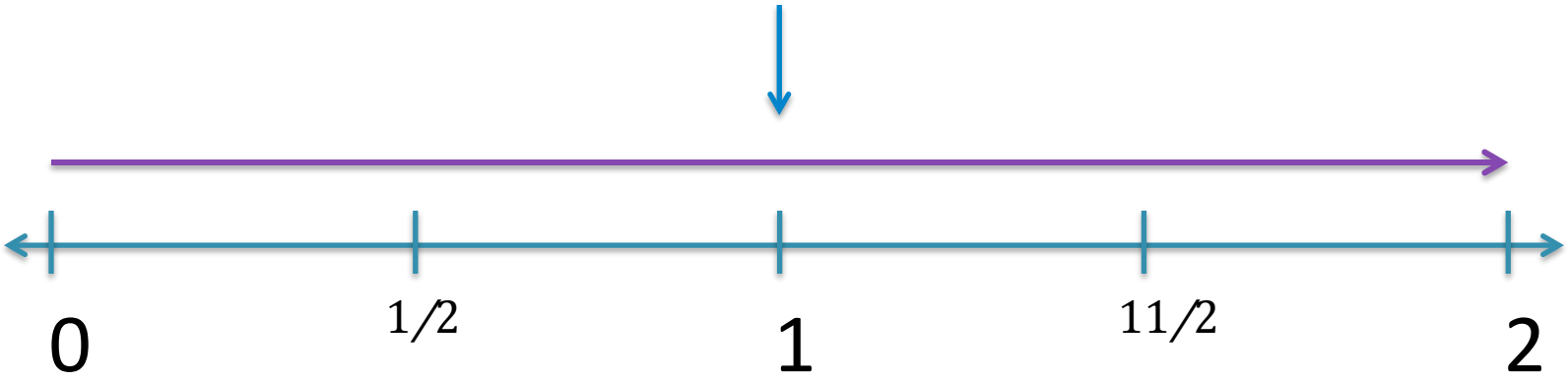
2 of $1/2$ or $1/2$ of 2



2 *of* 1/2

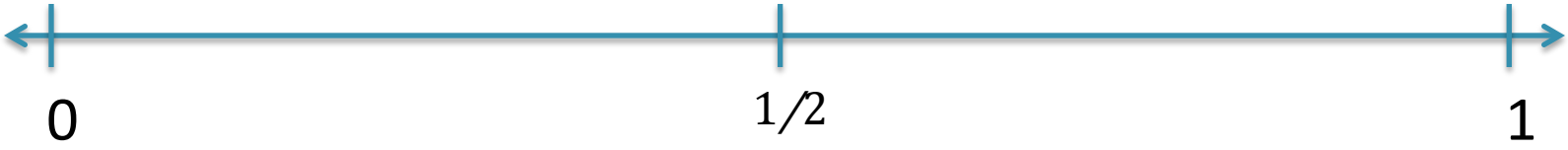


1/2 of 2

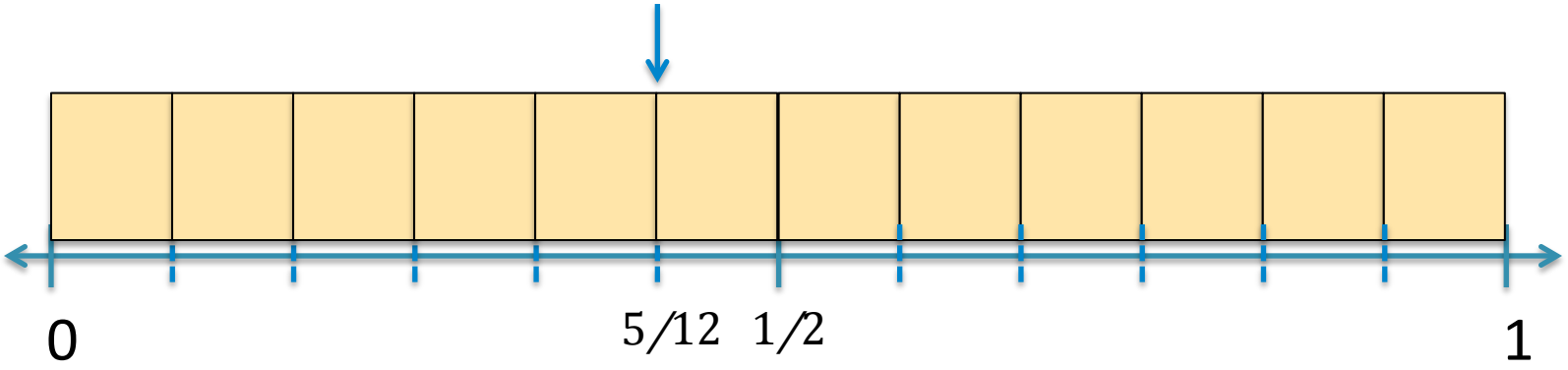


$5/6 \times 1/2$

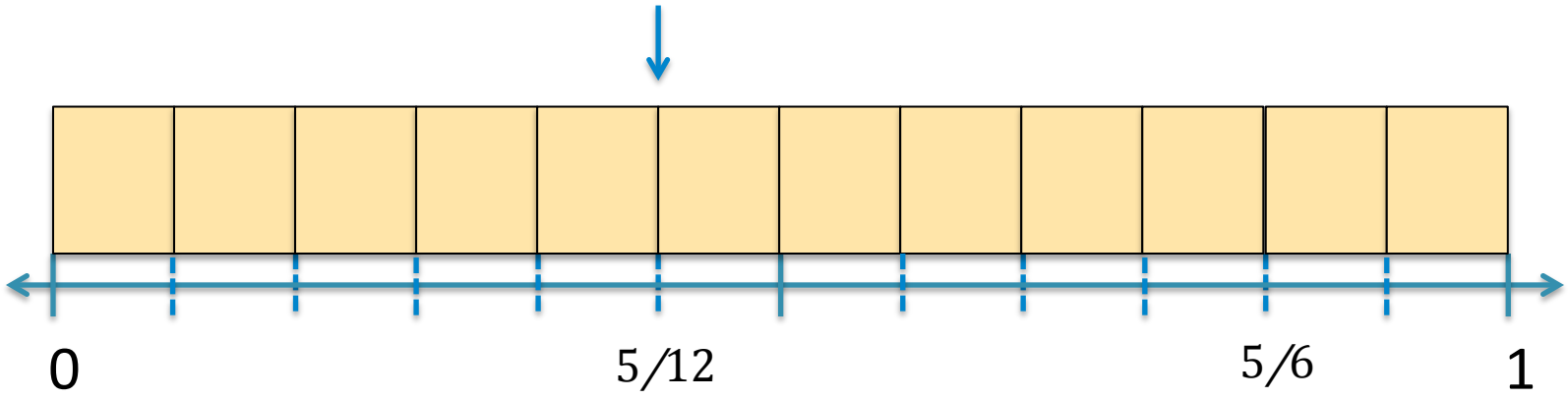
$5/6$ of $1/2$ or $1/2$ of $5/6$



5/6 of 1/2



1/2 *of* 5/6



Fraction Sense and Division

Yours is not to reason why;
just invert and multiply.

Consider.....

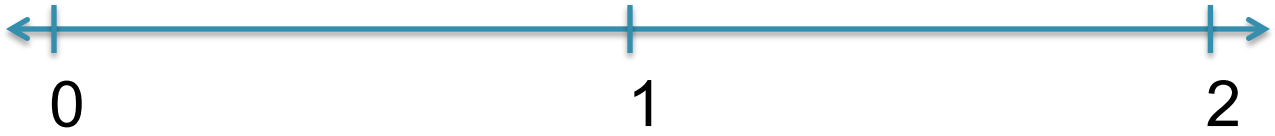
- $6 \div 2$: How many 2s are in 6?
- $6 \div 1$: How many 1s are in 6?
- $6 \div 1/2$: How many $1/2$ s are in 6?
- $6 \div 2/3$: How many $2/3$ s are in 6?
- $2/3 \div 1/2$: How many $1/2$ s are in $2/3$?

NAEP, 2003 (Modified)

Jim has $5/6$ of a yard of string which he wishes to divide into pieces, each $1/12$ of a yard long. How many pieces will he have?

How many $\frac{1}{12}$s of a yard are in $\frac{5}{6}$ of a yard?

2-Unit Number Line



How many $\frac{1}{12}$ *s of a yard are in* $\frac{5}{6}$ *of a yard?*

Division on the Number Line

Division Practice Problems

Use the rods and your number line to solve the following problems:

$\frac{2}{3} \mid \frac{1}{6}$

$1\frac{2}{3} \mid \frac{1}{2}$

$\frac{5}{6} \mid \frac{1}{3}$

$\frac{1}{4} \mid \frac{3}{4}$

Something else to consider…

- You have 28 students in your class.
- You want to split them into 2 equal groups.
  - Use words to describe how you did this.
  - Write an equation to describe how you did this.

$$28 \div 2 = 14$$

$$28 \times 1/2 = 14$$

Inverse Operations

Addition

A number's *additive inverse* is the number that you can add to the original number to get the additive identity (0). For example, the additive inverse of 67 is -67, because 67 + -67 = 0, the additive identity.

Adding a number is the same as subtracting its inverse.

$$6 + {}^{-}3 = 6 - 3$$

Inverse Operations

Multiplication

A number's *multiplicative inverse* is the number that you multiply the original number by to get the multiplicative identity (1).

Since 6 x $1/6$ = 1 the multiplicative inverse of 6 is $1/6$.

Dividing by a number is the same as multiplying by its inverse.

$$6 \div 2 = 6 \times 1/2$$

Invert and Multiply

$$\frac{4}{15} \div \frac{2}{3}$$

Invert and Multiply

$$\frac{3}{2} \times \frac{4}{15} \div \frac{2}{3} \times \frac{3}{2}$$

Invert and Multiply

$$\frac{3}{2} \times \frac{4}{15}$$

In order to understand this….

Students need to understand this…

$$24 \div 6$$

$$2 \times 24 \div 2 \times 6$$

$$\tfrac{1}{2} \times 24 \div \tfrac{1}{2} \times 6$$

Multiplying or dividing the dividend and divisor by the same number will not change the quotient.

In addition – you really can divide across numerators and denominators…

$4/15 \div 2/3$

Sometimes it's more elegant than others....

$4/15 \div 2/3$

$$= \frac{2}{5}$$

Sometimes, well not so much…

$$4/15 \div 3/2 = \frac{4/3}{15/2}$$

Another Proof

$$4/15 \div 3/2 = \frac{4/3}{15/2}$$

Another Proof

$$\frac{4}{15} \div \frac{3}{2} = \frac{\frac{4}{15}}{\frac{3}{2}} \times \frac{\frac{2}{3}}{\frac{2}{3}}$$

What is Fraction Sense?

Fraction sense is tied to common sense: Students with fraction sense can reason about fractions and don't apply rules and procedures blindly - nor do they give nonsensical answers to problems involving fractions.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

What can you do to help?

- Support students' understanding of fractions as numbers.
- Support students' understanding of equivalence.
- Focus on properties of arithmetic.
- Help students appropriately connect whole-number knowledge to working with fractions.
- Provide opportunities for students to reason about fractions and share their thinking.

"Children who are successful at making sense of mathematics are those who believe that mathematics makes sense."

-Lauren Resnick

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Thank you!!

jmcnamara@mathsolutions.com

mathsolutions.com
800.868.9092
info@mathsolutions.com