7

FINDING THE AREA OF A CIRCLE

This unit was developed from an article that appeared in the May 1986 issue of *Arithmetic Teacher*, "Finding the Area of a Circle: Use a Cake Pan and Leave Out the Pi," by Walter Szetela and Douglas T. Owens. The article presents a collection of methods for approximating the area of a circle, a topic usually taught in the middle-school curriculum.

Traditionally, students learn the formula $A = \pi r^2$ and demonstrate mastery by plugging in numbers to get answers. Though teachers may have explained the reason for the formula, it's not unusual for students to calculate areas without knowing why the formula makes sense. As stated in the article, "Unfortunately, numbers, and not understanding, are all that result."

The *Arithmetic Teacher* article offers a different slant. It provides a collection of methods for approximating the area of a circular region. These methods build on methods for measuring the areas of triangles, squares, rectangles, and parallelograms. Though the activities admittedly take more instructional time than merely teaching the formula, they do much more. They promote understanding of the idea of area in general and, more specifically, of the idea of finding the area of circles.

Six of the methods from the article were adapted for this unit. Along with explaining the math explorations, the chapter describes how the explorations were structured into a unit of study and provides details about how the class was organized. Information is included about the materials needed, how the methods were translated into a menu of tasks for the students, how the students' recording was organized, what assessments were done, and how students' work was graded.

The unit was presented to a class of sixth graders and a class of seventh and eighth graders. In both classes, the students worked in pairs. They worked at their own pace and made their own decisions about the sequence in which to do the tasks. The chapter describes in detail what happened in the sixth-grade class and includes a note about some of the differences experienced with the seventh and eighth graders.

Getting Ready

To prepare for the unit, I drew a circle with a radius of 10 centimeters on a piece of paper and duplicated the paper about 60 times. I also drew a 10-centimeter circle on a sheet ruled into square centimeters and duplicated about 30 sheets. This was sufficient for the students to use for all the methods.

By using the same-sized circle to test each of the six methods, the students could compare the approximations of its area that each method produced. Also, because everyone in the class was using the same-sized circle, students could compare their results.

I typed the directions for the six methods to distribute to the students and also prepared a cover sheet of instructions (see p. 112). Even though I planned to introduce the investigations verbally to the class, I know it's helpful for students to have a written reference for clarification.

In addition, I assembled the materials needed. I filled four small plastic bags with lima beans, enough in each to cover the circular region. I bought some pliable linoleum floor covering and cut two circles the same size as the ones I duplicated, six 10-by-10-centimeter squares, ten 10-by-1-centimeter strips, and twenty 1-by-1-centimeter squares. I divided the linoleum pieces equally into two plastic bags. (My local lumberyard sells floor covering in 12-foot rolls. I bought a 1-foot strip, which was about 3 times what I needed. The linoleum was a bargain, as it was on sale for $4.77 a yard.) I set out one pan balance. As for all lessons, rulers, scissors, tape, and calculators were also available for the students.

Discussing Area

I organized students into pairs for the investigation. Before giving them the written instructions and explaining the materials I had assembled, I decided to review what the class had previously studied about area as a way of leading into the new activities.

"What's meant by *area*?" I said, and wrote "area" on the board.

Several students had ideas.

"It's the distance inside something," Chris offered.

"It's the space inside a shape," Meghan added.

"Yeah," Chris blurted out, "it's space, not distance. That's what I meant."

"You do it in square feet," Amir said.

"It has to do with measuring space that's inside a closed figure," Michelle said.

"It's the space inside the perimeter," Nick said.

After all the students who wanted to contribute had spoken, I drew a rectangle on the board. "How would you find the area of this rectangle?" I asked.

"You multiply length times width," Jason and Cory called out, almost simultaneously. Others nodded their assent.

"What does that tell you?" I asked. "Raise your hands if you can explain what multiplying the length by the width tells."

I told the students to raise their hands to discourage them from calling out before being recognized. Also, I wanted to allow time for all students to think about my question. I waited a bit to see which students would decide they could explain this. About a third of the students raised their hands. I called on Meghan.

"It tells you the area," she said. "Like if the length is 8 and the width is 5, then the area is 40."

"How could I show the 8, 5, and 40 on my drawing?" I asked.

Nate answered. "You just draw lines to make squares," he said. "You show 8 squares one way and 5 squares the other."

I divided the rectangle into squares as Nate suggested and said, "Drawing the squares this way may help explain to someone why a rectangle with a length of 8 units and a width of 5 units has an area of 40 square units."

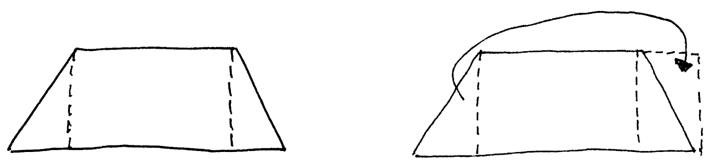
I find that students continually fail to include the labels for units, especially for square units, in spite of my pointing this out repeatedly. I try to draw attention to the correct labels as much as possible, using the correct mathematical language as often as I can.

I drew several other rectangles on the board, including one square, showing how each could be divided into squares to find its area. Then I drew a trapezoid the shape of the red trapezoid from the Pattern Blocks.

"How would you find the area of this trapezoid?" I asked.

After a few moments, two hands went up. I called on Jeff.

"You could draw lines straight down so you have a rectangle in the middle," he suggested. I did this.



"Oh, I see," Jennifer said. "You could put the triangle from one side on the other so it would be a rectangle. Then it's easy."

I sketched this and commented, "What this does is change a new problem into one you already know how to solve. That's a useful problem-solving strategy and one that works for other shapes as well."

I drew a parallelogram on the board. "Jeff and Jennifer's idea would also work for this parallelogram," I said, illustrating this on the board.

"Other shapes might be more complicated," I said, drawing on the board a right triangle, a hexagon, and an irregular polygon with eight sides. "But you could use what you know about rectangles by cutting and piecing these together to figure their areas."

Matt raised his hand to contribute something he remembered. "The triangle is easy," he said. "You just make it into a rectangle and then cut your answer in half."

From this discussion, I learned that the students knew something about area, though individuals had different bits of information and understanding. Some seemed clear about what area is and how to find it while others were shaky in their knowledge. I wasn't sure about those who didn't volunteer ideas.

Focusing on Circles

I then drew a circle on the board. "What about finding the area inside a circle?" I asked.

FINDING THE AREA OF A CIRCLE

Names ______________________________

This investigation presents six different ways to approximate the area of a circular region. Following are your jobs for this unit.

1. WHAT TO DO

   Working together as partners, try each method. Make sure you agree on your results.

2. HOW TO RECORD

   Getting an approximation is only one part of the goal for each method. Also important is to analyze the mathematics of the method -- why it makes sense, how accurate it is, and whether it's a reasonable approach. Working together, use the following format and write about each method:

   > Method No. - Title
   >
   > Description: (Tell what you did.)
   >
   > Result: (Record your approximation.)
   >
   > Analysis: (Describe why the method makes mathematical sense. Also describe how you feel about the accuracy of the method. Include other reactions you have.)

3. REFLECTING ON YOUR EXPERIENCE

   When you and your partner have completed all the methods, individually answer the following questions in writing:

   Which method do you "trust" the most? Why?

4. EXTRA (Optional)

   Invent another method for approximating the area of a circular region. Try it. Describe your method and the results.

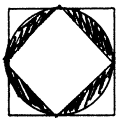
Note: When you've completed the activities, organize your work for each method. Make a cover sheet that has a title, date, your names and include a table of contents for what you're submitting.

FINDING THE AREA OF A CIRCLE

Method 1: COUNTING SQUARES

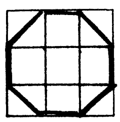
1. Using a circle drawn on squared centimeter paper, count all the whole centimeter squares that lie completely inside the circle. (This underestimates the area of the circle.)
2. Now make an overestimate of the area of the circle by taking the number of whole squares that lie inside the circle (the same number you got for step 1) and add to it the number of squares that touch the circle and lie partly inside and partly outside.
3. Average the two counts.

Method 2: INSCRIBING AND CIRCUMSCRIBING SQUARES



1. Circumscribe a square about the circle. Find its area.
2. Inscribe a square inside the circle. Find its area.
3. Average the two areas.

Method 3: THE OCTAGONAL (OR EGYPTIAN) METHOD

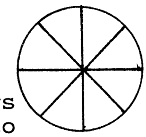


1. Circumscribe a square about the circle. Find its area.
2. Divide the square into 9 congruent squares.
3. Form an octagon by drawing a diagonal in each corner square as shown.
4. Figure the area of the octagon to approximate the area of the circle.

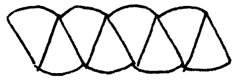
Method 4: FINDING THE AREA BY WEIGHING

1. Weigh the circular tile cutout with the rectangular tile pieces.
2. Use the rectangular pieces to approximate the area of the circular region.

Method 5: THE "CURVY PARALLELOGRAM" METHOD



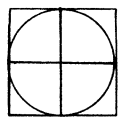
1. Divide the circle into 8 congruent sectors.
2. Cut out the sectors and arrange them to form a curvy parallelogram.



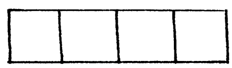
3. Approximate the area of the curvy parallelogram.

Method 6: USING BEANS

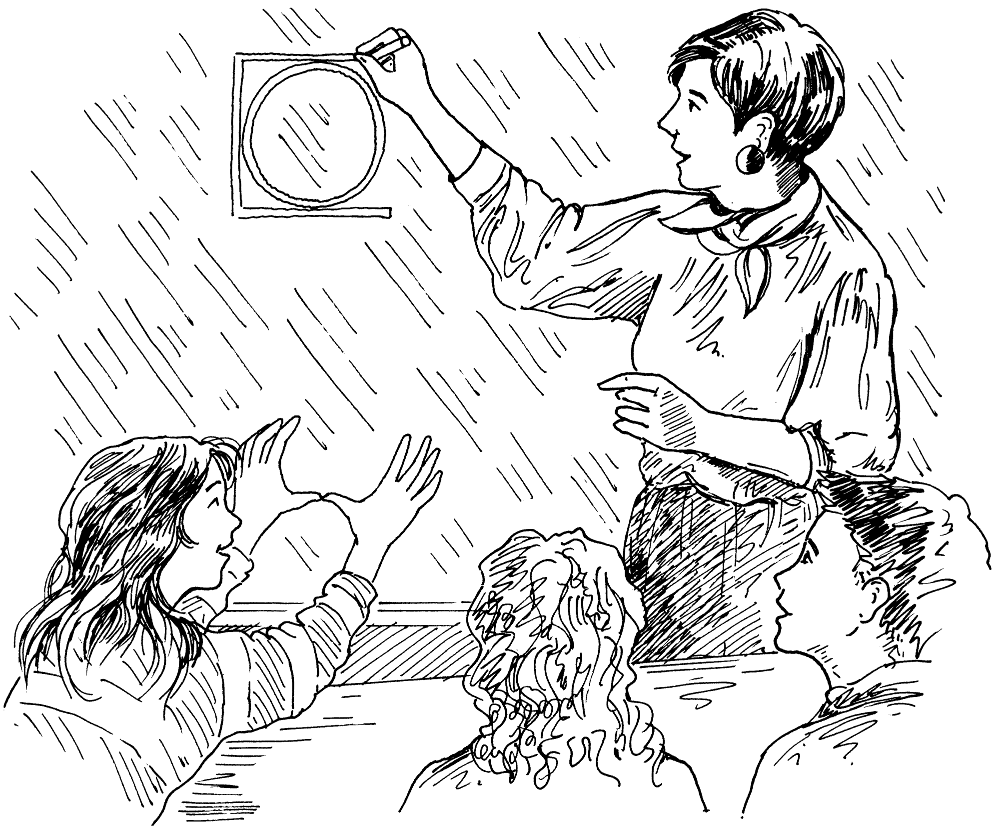
1. Cover the circular region with one layer of beans. (It's helpful to use a collar to "corral" the beans.)
2. Circumscribe a square about the circle. Cut it into four smaller squares as shown.



3. Rearrange the four squares into a rectangle. Tape the pieces together.



4. Place the beans on the rectangle you've made, pushing them toward one end.
5. Use the area covered by the beans to approximate the area of the circle.



Jennifer suggests drawing a square around the circle to approximate its area.

"You need the formula," Amir volunteered. No one knew or could recall what it was.

"Yes," I answered, "there is a formula. But what could you do to solve the problem of finding the area without a formula?"

Meghan raised her hand. "You could cut it apart and piece it together into a rectangle," she said.

"That would be hard with all the curves," Chris reacted.

"I have an idea," Jennifer said. "Draw a square around the circle." I drew the square.

"Now what?" I asked.

"You can find the area of the square," Jennifer continued. "But that would be too big, so you'd kind of have to figure how much those extra parts were and subtract them."

Some of the others seemed impressed with Jennifer's suggestion. I was impressed also, as Jennifer's idea was close to what the class would do in one of the methods they were going to use.

"Does anyone have another idea?" I asked.

"You could draw a square inside instead," Michael said, "and then figure how much extra you need to add."

I drew a square inside the circle to illustrate Michael's idea. "There are special words that describe the squares that Jennifer and Michael described," I told the class. I wrote "circumscribed" and "inscribed" under "area" on the board.

"Which word describes Jennifer's square, and which describes Michael's?" I asked. It was easy for the class to decide. I then began an explanation of what they would do in this investigation.

"I've drawn the same-sized circle on plain paper and on squared-centimeter paper," I said to the class. I held up a copy of each for them to see.

"In this unit," I continued, "you're going to approximate the area of this circle in six different ways."

"I have another idea," Chris said. "You could count all the whole squares and then figure out how to put the little pieces of squares together to find how many total squares there are."

"That's how I could figure how many squares are in the extra pieces," Jennifer said, referring back to her idea.

"Remember those ideas," I said. "The ones I'm going to describe to you are different from your ideas, but you'll have the chance to try your own method after you've tried the ones I've prepared. Let me explain one method you'll try," I said.

"Should we take notes?" Nell asked, always organized.

"You don't have to," I said. "After I describe a few of the methods, I'll pass out complete instructions, and we'll go over them all together."

I decided to present two methods verbally first, without their having the distraction of looking at papers as well as listening. Then I'd have them read the descriptions to reinforce what I presented and give further directions.

"One bit of useful information," I said, "is that the circle you'll be measuring has a radius of 10 centimeters." I added "radius" to the list of words I'd already started on the board. I wrote "diameter" underneath.

"How long is the diameter?" I asked.

I received a chorus of "20" in reply. Most of them seemed to know the diameter was twice the radius. I drew a circle on the board and labeled the radius and diameter.

"One method you'll use is a combination of what Jennifer and Michael suggested," I said.

I drew a circle on the board and circumscribed a square around it. "First you circumscribe a square as Jennifer did," I said. "Then you figure its area. How would you do that?"

"If the circle were on the yellow paper," Lauren said, "you could count the small squares." The yellow paper was the one on which the circle was drawn over squared centimeters.

"You could measure the length and width and times it," Keky said.

"You just have to measure one side because it's a square," Nell said.

None of the students related the radius or diameter of the circle to the side of the square. I showed on the board that the side of the square was equal to the diameter, but I know that students only use methods that they understand and are comfortable with.

"The area of the circumscribed square overestimates the area inside the circle," I said. "Who can explain what I mean by that?"

This seemed obvious to the class. About a third of the students raised their hands. Josh explained.

"That's the first part of this method, getting an approximation that's too big," I continued. "Then you inscribe a square, as Michael suggested, and figure that area, which will be too small an estimate. The final part of the method tells you to come up with an approximation between these two areas by averaging."

There was a chorus of murmurs from the class: "Oh, that's neat." "I get it." "Nifty."

Though I wasn't sure they all understood averaging, I decided to have them face that with their partners and ask me later if needed.

"The method I just explained is number 2 on the sheet I'll pass out in a moment," I said. "Method number 1 also has you average but in another way."

I held up a circle on the squared-centimeter paper.

"In this method, you again overestimate and underestimate and average the two," I explained. "To underestimate, you make a count of all the whole squares that lie inside the circle. To overestimate, you make a count of all the squares inside or touching the circle, whether or not they're whole squares. In this second count, some of the squares you include are on the circle, with part inside and part outside. Then you average these two counts."

I continued now with instructions for how I expected them to record. "For each of the six methods you try," I said, "you and your partner are to make a record using a separate sheet of paper for each method. There are four parts to what you write on each sheet: title, description, result, and analysis."

I drew on the board a facsimile of a sheet of paper and showed how I wanted it structured. I gave further directions. "Copy the method number and title from your sheet of directions," I said. "For the description, tell what you did. Then record the approximation you got. Be sure to label your answer." I wrote "square centimeters," "sq cm," and "cm^2" on the board to give them options for labeling their answers.

"The analysis is the part of your recording," I continued, "where you describe why the method makes mathematical sense and how you feel about the accuracy of the method. Also, you can include any other reactions or thoughts you have in this section."

I added one comment before passing out the written directions. "You need to have one record sheet for each method for the two of you," I said, "and I'd like you to discuss your ideas before either of you begins to write."

Then I passed out the two direction sheets, which I had stapled (see illustrations). "Put your names on the first sheet. Together, read what I've written before I give you more information."

I gave them time to look over what I had prepared. I realized that this was a lot of information for them to absorb, but I've learned from experience that it's worth the effort to give directions as completely as possible before students plunge into work. After they had had time to read the material, I called them back to attention.

"I'd like to talk a bit more about the other methods," I said, "and show you the materials you'll be using. I've described the first two. For number 3, use the circle on the plain paper. It might be helpful to make a note of this on your instructions." I waited for them to do this.

"Again, circumscribe a square about the circle," I continued. "Then divide this square into nine congruent squares, and draw diagonals as shown to form an octagon." I added "congruent," "diagonal," and "octagon" to the list on the board and had students explain each.

"Then you figure the area of the octagon and use that as an approximation of the area of the circle," I said.

"How do we do that?" Amir asked.

"I won't answer that just now," I responded. "Try and figure it out when you get to it. I'll help if you and your partner are stuck."

"For number 4," I continued, "you'll need to use one of the plastic bags of linoleum pieces and the pan balance. Weigh the circular tile, which is the same size as the circle on the paper, with the rectangular pieces. Use the information you get to approximate the area of the circular region."

There were no questions. I knew they were anxious for me to stop talking so they could get started. But I pursued the directions for the last two methods.

"For method 5," I said, "make a note to use a circle on plain paper. Cut out the circle and then cut it into eight sectors as if you were cutting a pizza. Arrange the eight sectors to make a 'curvy parallelogram,' as I drew." I added "sector" to the list of words.

"What's a *curvy parallelogram*?" Keky asked.

"It looks like a parallelogram," I said, "but the two long sides aren't straight. They're pieces of the circumference."

"How do we find its area?" Keky pursued.

"I know," Michael said, "you do length times width like with the other parallelograms."

"These are the things you'll need to discuss with your partner when you work," I said.

"One more method," I continued. "Take a plastic bag of beans for this and cover a circle, patting the beans carefully into one layer."

"Which circle do we use?" Nell asked.

"Either will do," I said. "It's up to you and your partner to decide. Then draw a circumscribed square, cut it into fourths, tape the squares into a rectangle, and rearrange the beans to see how much of the rectangle they cover. Use this to approximate the area of the circle."

I gave a few last directions before letting them get to work. "You can try the methods in any order you like," I said. "Decide with your partner which you're interested in trying first. Only one pair can do method 4 at a time, though, since we only have one pan balance. Any questions?"

There were none. At this time, only fifteen minutes remained in the period, and they all began work. I circulated, answering questions. "Which circle should we use for method 1?" "Can we work on separate ones?" "Do we each have to record?" "Which is an easy one?" "I don't get number 5." "What do I write?"

I answered their questions as briefly as possible, encouraging them to work through their initial confusion with their partners. I planned to talk with the whole class at the beginning of the next period about some of the questions they raised, but wanted them to all have some beginning experience. Their confusion was typical for the beginning of a unit such as this. I've come to learn that students settle down once they feel more comfortable with what they're expected to do.

Beginning the Second Day

Before having the students begin working on the second day, I talked with them first. I wanted both to answer the questions they had and to bring attention to some issues of concern to me.

"Before you get back to work on the circle activities," I said, "let me have your attention for a few minutes." The students were in various stages of getting settled and organized for work.

"What do I do?" Melissa asked. She had been absent the day before.

"Me, too," Gabe said.

I settled the class down. "Who can describe for Melissa and Gabe what you began working on yesterday?" I asked.

"We have these circles," Cory said, "and we have to figure out their area in a bunch of different ways. Each of the ways is on this sheet." Cory held up her sheet of instructions for Melissa and Gabe.

"Does anyone have anything else to add?" I asked.

Keky raised her hand. "You use the yellow sheet for some and the white sheet for others depending on whether you'll need the little squares or not." She held up a sample of each and continued, "Like for the one with counting squares you have to have the squares, but you don't need them for all the others."

"You get your answers in square centimeters," Michael added, "and you can write that three different ways. See up there on the board?" Michael pointed to where I had written "square centimeters," "sq cm," and "cm^2."

"All the answers are approximations," Chris said.

It was interesting for me to hear what the students had to offer in their descriptions. It gave me feedback on how they had interpreted the directions I had given.

"Can Gabe work with us?" Jeff asked.

"No, Melissa and Gabe can be partners together," I said. I had them move so they were seated together. "I'll help you get started once the class gets to work," I told them.

"I want to talk a little bit about the writing I've asked you to do for each method," I then said to the class. "I read a book last summer titled *Writing to Learn* written by William Zinsser. The book gave me a great deal to think about, and one quote from the book has stuck in my mind: 'Writing is how we think our way into a subject and make it our own.' "

I repeated the quote for the class (Zinsser 1988, 16) and then asked, "What do you think Mr. Zinsser meant when he wrote that?"

Several of the students had ideas to share.

"When you write something down in your own words, then you really understand it," Chris said.

"Your own words belong to you," Keky said, "and they're your ideas and no one can change them."

"You have to think about something before you can explain what you think in writing," Meghan said.

"When you have to write something," Nate said, "then you have to organize your thoughts. You can't just write something down; you have to think it through."

"I don't know how to say this," Amir began, "but when you have to write, first you have to figure out what you're going to write or else you can't write it."

"Writing forces you to think," Michael said.

"All of your ideas explain how writing is a way for you to get your thinking down on paper," I said after all the students who raised their hands had had a chance to speak. "You expressed your thoughts in your own words. Writing down those thoughts would be a way to think even further. I believe that having you write encourages you to think more fully about the mathematics you are doing."

I then continued with another reason for writing. "Not only does writing help you think about the mathematics you're exploring," I said, "but it helps me as a teacher, too. When we have a class discussion, not everyone gets the chance to say what they think. But it's important for me to know what each of you is thinking and to learn as much as possible about what you're learning and understanding. Your writing gives me insights into each of your ideas. So I'd like the recording you do to be as thorough and thoughtful as you can make it."

Meghan raised her hand. "Sometimes I don't know what to write."

"Writing is an extension of your thinking," I answered. "Sometimes it's easier to say your ideas than write them. That's where partners can help each other. Discuss what you think with each other to help you form your ideas into words you can write down. If you're both stuck, then raise your hands, and I'll help you start discussing what you've done."

"Is it OK if we divide the work so we do different methods?" Gabe asked.

"It's all right to share the work," I answered. "But what's important is that you each understand each method well enough so you could explain it to someone else. Also, even though only one of you will do the actual writing for a method, I want you first to discuss together what you're going to write. Explore your ideas together and then write as detailed a report as possible to help me understand your thinking."

"What do you do if you get different answers?" Keky asked.

"You most likely will get different answers for the different methods because they're all approximations," I answered.

"That's not what I meant," Keky went on. "Suppose you and your partner both do the same method and get different answers."

"That could happen as well," I said. "Try and identify why they're different. You may be able to come to an agreement on one answer or you may need to report both results in your record."

This was a good time to have them begin working. "Continue where you left off yesterday," I said. "Remember, you can do the methods in any order you like, but be sure to take special care with the write-ups."

Annette raised her hand. "Jennifer's absent," she said. "What should I do?"

I called Annette, Melissa, and Gabe to the front of the room. First I answered Annette. "Continue working by yourself," I said. "Then when Jennifer returns, you'll need to explain to her all that you've done."

Annette nodded and returned to her seat.

Then I went through an introduction of the methods for Melissa and Gabe. Rather than go through all six methods, I explained just the first one and told them to complete it and then call me over for further instructions. This gave me the chance to get back to supervising the rest of the class.

Observing and Helping Students

Students worked in different ways. Some pairs worked together on each method. Some tackled different ones. While some wrote descriptions after doing each method, others preferred to do all the methods first and the writing later.

When students asked me for help or reassurance that they were on the right track, my first response was to see if partners had checked with each other. If not, I would have them talk with each other and call me back if they were still stuck. If partners both had the same question, I would listen to what they had done so far and then give them feedback.

"We don't know what to do with the octagon," Jeff reported for him and Chris.

"What have you done so far?" I asked.

"We drew the square and divided it into 9 squares," Chris showed me. "Then we drew the lines to make an octagon."

"Now what?" I asked.

"We don't know," Jeff said.

"The method is based on the assumption," I explained, "that the octagon is pretty close in area to the circle." The boys nodded.

"You divided the circumscribed square into 9 squares," I continued. "How many of those 9 squares are taken up by the octagon?"

"Five," Jeff said.

"Yeah, but look," Chris added, "there are extra pieces. That's 5½, 6, 6½, 7. It's 7. Is that it?"

"I agree with that," I said, "but you'll have to convince Jeff as well."

"But then what?" Chris said.

"If you knew what the area of one of the small squares was," I said, "then you could find the area of the octagon."

"I get it now," Chris said.

"Explain what you understand to Jeff," I said. "I'll come back in a bit and check on how you're doing."

That seemed to be enough for the boys. When I checked back in a few minutes, I noticed they were using a ruler to measure the side of the smaller squares. As they were busily at work, I didn't interrupt them.

Meghan called me over. She and Cory had weighed the linoleum pieces and gotten an answer of 310 square centimeters. They were writing their report.

"I think this method works," she said, "because the square pieces really are the same amount as the circle, but they're just in different shapes."

"Yes," I answered. "What you describe makes sense to me, and that's just the kind of explanation you need to make in the analysis."

"Can we use a drawing?" Cory asked.

"Yes, that would be fine," I said. "Include whatever will help make your explanation clear."

Sometimes I'd notice students were on the wrong track. For example, Katie and Jason made an error with the first method. They counted the number of whole squares inside the circle and got an underestimate of 282. Then they counted just the squares touching the circle, getting a count of 58, instead of the overestimate they were supposed to figure, which included the first count as well. Their averaging produced an answer of 170.

"Something doesn't make sense here," I said to them. "What does 282 tell you?"

"It's how many squares are inside the circle," Jason answered.

"Is that number greater or less than the area of the circle?" I asked.

"Less," they both answered.

"Yes, it's an underestimate," I said. "But your final answer of 170 is even smaller. That doesn't make sense." I pointed to their work.

"Uh oh," Katie said, "we must have counted wrong."

"We couldn't have," Jason said, "we checked."

The students use different methods to approximate the area of the circle.

I reminded them about averaging an underestimate and overestimate to get an approximation of the area. "You didn't make an overestimate. An overestimate would include all the squares inside the circle, that you already counted, as well as the squares touching the circle. That would be 282 plus 58."

That was enough for them and they got back to work to revise their answer.

A few other pairs of students ran into the same problem. Rather than go through the explanation with them, I had them go and ask Jason and Katie for help. Explaining would help Jason and Katie cement their thinking. Also, it would keep me free to help others.

At this moment, Michelle and Gabe had completed the first methods and needed further directions. Also, Amir and Michael were stuck on the octagonal method. I had Amir and Michael get help from Chris and Jeff and went to talk with Michelle and Gabe.

There was a great deal of activity in the room. I was busy helping and directing. The class still didn't seem settled in the project, as students were still struggling with the unfamiliarity of what they had to do. But the students' activity seemed generally purposeful.

When talking with students, I continually reminded them that their job was to do what made mathematical sense to them. Getting right answers quickly to finish wasn't the goal; rather, investigating different mathematical ways to approach finding the area was most important. I also prodded them to describe fully what they did. Getting students to focus thoughtfully on their tasks and write clear and thorough explanations takes the time and effort of consistent reinforcement.

When only about four minutes remained in the period, I asked the students to get ready to leave. "Collect your work," I said, "and keep it together so you're ready to begin work when you come to class tomorrow.

The Next Few Days

When students arrived the third day, I had them get right to work. I wanted to institute the procedure that when they came to class, they were to get started immediately on their investigations. With periods only forty-three minutes long, it's important to make good use of all the time available.

Students became more self-sufficient in their work over the next few days. Partners devised their own systems for working. There was interaction among pairs as they checked with each other for advice.

While the students worked, I circulated. I answered questions, directed students who seemed to be wandering to get back to their task, observed what pairs were doing, and listened to students' discussions. This kind of setting in the classroom is enormously valuable for informally gathering information about what students do and think. It allows for observing and interacting with students in ways not possible during whole class lessons when the teacher is presenting information or leading a discussion.

It was interesting for me to notice the different approaches students used. For example, when students were trying the octagonal method, some measured the sides of the small squares as Chris and Jeff did. Their results differed depending on the accuracy of their drawing and measuring.

Other students, however, had different approaches. Marissa wrote the following for herself and Lauren: *I multiplied 20 × 20 to get the area of the square. I got 400. Then I divided 9 into 400 because there were 9 squares and by doing this I got the area of each square. Next I multiplied 7 × 44. I did this because 44 square centimeters are in each square and there are 7 squares in the octagon.* They had done their calculations with pencil and paper.

Bryan and Jon used the same method as Marissa and Lauren did but didn't round off. They had used a calculator. Jon explained how he drew the octagon and then continued to write: *I found how many littler squares there were, and I got seven. I found the area of one of them by dividing 400 by nine and I got* $44.\overline{44}$*. then I took* $44.\overline{44}$ *and multiplyed it by seven.* When results were presented the next week, Bryan explained that the line above the 4s indicated that it kept repeating. This was new notation for most of the students and a nice way for it to be introduced.

Michelle and Gabe used a different procedure. Gabe wrote: *We found the area of the octagon by finding the area of the whole big square and subtracting the part that wasn't also part of the octagon.* They had multiplied 44.44 by 2 and subtracted that from 400.

Cory and Meghan arrived at their result by multiplying 400 by $\frac{7}{9}$. Cory explained: *The reason why I × 400 by* $\frac{7}{9}$ *is because there are 9 squares alltogether and seven of them are in the octagon.*

When working on method 1, most students did what Nell and Keky described, although they arrived at different numerical counts: *We counted the squares that were completely inside the circle and got 284. We then counted the squres that had any bit of the circle inside them and got 344. We added 344 and 284 and divided by two to get our answer.*

Michael and Amir, however, used a shortcut. They wrote: *First, we counted the whole centimeter squares that lie completely in the circle. To make sure that we didn't lose our count we numbered inside the squares. After we had numbered half-way through the circle, we figured out that all we had to do was double the last number instead of numbering the rest of the circle.*

It took most of the students about five class periods to do all six methods and write their descriptions. Then they had to write individual reactions, describing the method they "trusted" the most and explaining why. Finally, they had to make a table of contents and organize and assemble their work into a booklet.

Organizing their work is a way to have students take another look at what they've done. Too often, when students finish their work, they never take a second look at it. They're no longer interested in examining what they did. Having them prepare what they've completed to submit encourages them to review their work.

Because some students always work more quickly than others, it's a good idea to have an extra optional activity. Though the students had the option of inventing another method for approximating the area inside the circle, none of them was interested in this. They were given another option on the fourth day. It was an activity called Half the Circumference. Using the circle they had been measuring as a basis, students were to draw a circle with a circumference half as long, approximate its area, and draw a conclusion about the relationship between the areas of the two circles. Three pairs of students did this task.

Reading and Evaluating the Students' Work

I read each of the students' booklets twice. My first reading gave me an overall sense of their responses. As I read through them, I made a chart of the approximations they reported.

Method	1	2	3	4	5	6
Josh and Jon	309	298	294	308	305	308
Bryan and Jon	305	310	311.11	312	310	320
Nell and Keky	313	298	316	306	303	missing
Michelle and Gabe	311	298	308	311	310	307
Lauren and Marissa	265.5	291	308	312	310	300
Jennifer and Annette	316	49 in^2	312	312	48 in^2	306
Chris and Jeff	310	312.5	304	311	310	311
Katie and Jason	312	305.5	410.8	312	300	316
Meghan and Cory	305	300	312	310	305	300
Amir and Michael	313	300	308	311	36 in^2	311
Graham and Jason	319	298	310.8	306	384	¾ of the rectangle

Even though I had observed the students working during class, I was surprised by what I read in some of their reports. For example, two pairs of students measured in inches for some of the methods, making comparison with centimeters difficult. Three pairs either neglected to include units in their reports or used incorrect labels. There was a missing task in one folder and missing individual reactions in two. One pair didn't write any analyses for their methods. They realized this, too late, and wrote an "Oops, sorry" apology.

There were differences in the contents of their reports in clarity, thoroughness, and mathematical insights. The greatest variation in quality occurred in what they included in their analyses.

Some of the students' analyses gave insights into their mathematical thinking. For example, one student gave the following explanation for the octagonal method: *I think this was a good method because the octagon is almost the same size as the circle so it gives almost acurate results.* Another student wrote: *This makes mathematical sense because if you look at the octagon and the circle the space they both occupie is incredibly similar.*

In contrast, other analyses expressed a minimum of mathematical interpretation. For the same method, a student wrote: *I feel that this answer is pretty accurate because I had to go through a lot of multiplying and adding ect. to get the answer.* Another wrote: *This method is good. It is accurate and takes only a short time. I would recommend it.*

Some of their writing revealed a struggle in their thinking. Also for the octagonal method, a boy wrote: *The only reason that this method works is that the octogon is shaped remotely like the circle. I don't feel that this method is very acurate because the shape of the octogon is too "squarey" to be acurate. Except for the fact that I tried this method, I wouldn't have thought that it would work. I really don't know why. It just*

Method Number One   Counting Squares

Description: The first thing we did was to count all of the whole squares inside the circle. We got 284. Then we counted all of the whole squares, not only inside the square, but touching, in any way, the outside of the circle. We got 344. We then averaged the two numbers.

Answer: 284+344=628   628÷2=314
314 square cm.

Analysis: I think this method is pretty accurate. I think this because the circle lies almost exactly in between the inside and outsid boundries that you make.

Jason and Katie clearly explained why method 1 made mathematical sense to them.

Method #2

Inscribing and Circumscribing Squares

First we circumscribed a square about the circle and found it's area. It was 400 square centimeters. Then we inscribed a square around the circle and found it's area. It was 211 square centimeters. Then we averaged the two areas. The answer we got was 305.5 sqaure centimeters.

This method sort of makes mathematical sense because when you circumscribe and inscribe the circle or square (whatever) you take some off then add it back on.

Brian and John weren't quite sure why method 2 made sense.

looks like the octogon is much too small. But I was wrong. (When I questioned what he meant by "this method works," he said that the answer he got was close to his other approximations.)

There was a similar range in the responses for other methods. For the analysis about the method that involved using beans, one student wrote: *It seemed to be a very odd method. The beans were fairly hard to work with, but I managed!* Another student wrote: *This experiment works because the area is shown by the same size units. By filling the circle with beans and using the same amount on a different figure, the area of beans will always equal the same area no matter what the shape is.*

I've come to expect such differences in the students' work. I know that students come to class with varying backgrounds of experience and different understandings. Partial understanding and confusion are natural aspects of the learning process. The more insights I have into students' thinking, the more able I am to make instructional choices that can add to their growing knowledge.

When reading the reports for a second time, I responded to each in writing. I planned to return the reports with my comments and give the students the opportunity to improve upon their work before I gave final grades. I feel that written feedback lets the students know that I have read and thought about their work. It helps them learn more about my expectations and standards. Also, when they read my comments, students have another chance to reflect on what they've done. Though grading is necessary, I'm interested in keeping the focus on their exploring and thinking as much as possible. The following are samples of what I wrote for several students.

Bryan and Jon "The descriptions you wrote for the six methods were clear. However, when you gave your reactions to each method, you didn't include very much about your *reasons*. It's your thinking that I'm interested in.

Method #3 - The Octagonal Method

Description - I first made a circumsived circle and then filled in the square with an octgon. after I did that I saw than five hole squares and four halfs of a sqaure. Then I saw that each square was 44.4 cm and so I just multiplid and got the ansoer.

Results - 310.8 cm - approximation

analysis - I feel that this answer is pretty accurate because I had to go through a lot of multiplying and adding ect. to get the answer. It makes sense becauso all squares ecqual the numler of centimeters multipled by number of squares

Jason and Graham were satisfied that the multiplying and adding they did insured the accuracy of their answer.

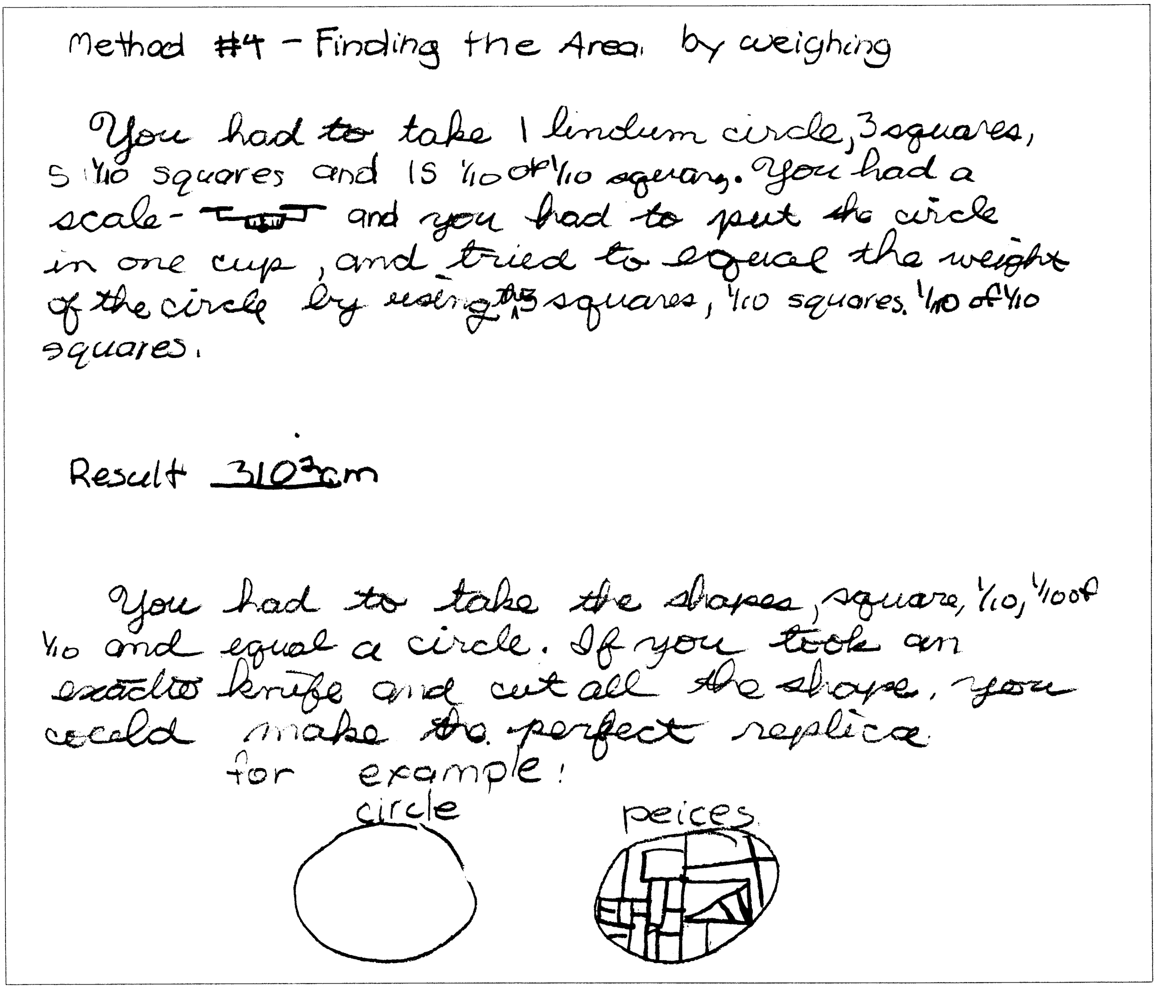
"Look at the reactions you each wrote for Counting Squares, for example. Bryan, you wrote that it was a good estimate but you didn't think it was effective. Why wasn't it effective? And Jon, you said you thought it was one of the more accurate methods. Why did you think that? Please rethink your reactions and add to what you've already written.

"Also, there are a few things missing:

1. Table of Contents
2. An individual paper from each of you that answers the following: Which method do you 'trust' the most? Why?

"P.S. I enjoyed your cover drawing."

Nell and Keky "Most of the descriptions you wrote for each of the six methods were clear, and your unit was presented well. I have some questions, however, about some of your work. I'd like you to consider and respond to my questions. You can write your responses on a separate sheet of paper or add them to the appropriate sheets in your folder.


Method #4 - Finding the Area by weighing

You had to take 1 lindum circle, 3 squares, 5 1/10 squares and 15 1/10 of 1/10 squares. You had a scale - and you had to put the circle in one cup, and tried to equal the weight of the circle by using the 3 squares, 1/10 squares, 1/10 of 1/10 squares.

Result 310^{2}cm

You had to take the shapes, square, 1/10, 1/100 of 1/10 and equal a circle. If you took an exacto knife and cut all the shape, you could make the perfect replica.
for example!

Meghan and Cory illustrated their reasoning about method number 4.

"Method 1: You said the method got you close to the initial answer. What do you mean by 'initial answer'? Please explain.

"Method 3: This method did not call for averaging. It called for figuring out the area inside the octagon by using the 9 smaller squares you drew. Try to do this and explain the answer you get.

"Method 5: How did you figure out the area of the curvy parallelogram? Please explain.

"Still missing from your project are the following:

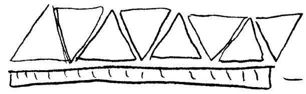
1. Method 6
2. Your individual reactions. This is to be an individual paper from each of you that answers the following: Which method do you 'trust' the most? Why?"

Amir and Michael "I enjoyed reading your work. What you wrote was descriptive and gave me good information about your thinking.

"Notice that for method 5, you measured in inches rather than in centimeters. This makes it difficult to compare the answer with the other methods. Please change your answer."

Method #5 The Curvy Paralelogram Method

Description: I used the white paper and cut out the circle. Then I folded it so it had eight sections. I cut out all eight sections and made a paraletogram out of them, like this:



—then, I meausured the parrelogram and found the approximate area of it.

Results: The area of the parrelogram was 310. So, the circle will be around 310.

Analysis: I found that this was a very good method, but it might not be right because the parrelogram was very curvy and so the meausurments might be a little messed up. I have gotten three answers that are practically the same though, 308, 310 (this one), and 312 (the wieghts).

Melissa and Lauren's analysis revealed their skepticism about method 5.

Summarizing the Unit with the Class

To begin a class discussion, I wrote on the board the students' results for the first method. I didn't include the students' names because I wanted them to focus on the mathematics, not on who got which answers.

Method 1

309	310
305	312
313	305
311	313
265.5	319
316	

Method #6 Using Beans

Description: I started off by filling in the circle with beans. Then, I cut the circle into four smaller square pieces. I organized the pieces into a rectangle and taped them together. I then began to fill in as many of the squares as I could with the beans. 3 of the squares were covered completely. I still had a few left over beans.

Result: The result is pretty much an educated estimate. The beans had covered 3 of the 4 portions of the rectangle. Since each of the small squares equal 100 of the 1 centimeter² boxes, 300 of the centimeter squares were covered. Yet, the extra beans were also lined up on another row. Our estimated guess seems to be very close to being exact. We got 311 approximately covered centimeter squares.

Analysis: At first, I was very confused. But when I went over the instructions again, it was all very clear. It seemed to be a very odd method. The beans were fairly hard to work with, but I managed!

Michael and Amir seemed satisfied with their effort for method 6.

"What do you think about the range of results?" I asked.

"They're pretty close," Michael said.

"Except for the one that's under 300," Nate added.

"Maybe they counted wrong," Keky said.

I posed another question. "How come there are different answers when you all used the same circle on the same squared-centimeter paper?" I asked.

"Well, it was hard to tell if some squares were really touching or not," Melissa said, "so some people could have included some that others didn't."

"They're just approximations," Cory said.

"You could make a mistake counting," Chris said. "There's a lot of counting."

"What ideas do you have about why the method called for taking the average of the two counts?" I then asked.

"You add them and then divide by 2," Keky said.

"I agree that that's how to find the average," I said, "but what does finding the average tell?"

Too often, students learn how to do a procedure without learning what the procedure accomplishes or why it makes sense. I waited to give the students time to think about my question. Not many volunteered to respond.

"It gives you a better answer," Lauren said.

"When you average, you get something in between," Amir said.

"Yeah," Chris added, "your answer is bigger than the first and smaller than the second."

"Because the first is too small and the second is too much," Nell said. "The average is about right."

I gave all the students who had an idea the chance to speak and then posed another question.

"Do you think this was a good method for finding the area?" I asked.

"I think counting all the squares made it very accurate," Jessica said.

"It was neat," Jason said. "I like the way it worked."

"It was too hard," Jeff said. "There was too much counting."

"Michael and Amir used a shortcut for counting," I said. "Could one of you describe what you did?"

Michael volunteered. He explained, "We only counted squares for half the circle and multiplied by 2." There were some murmurs of respect for his explanation.

I then changed the focus to the second method of circumscribing and inscribing squares.

"Let's take a look at the approximations you got for the next method," I said. I wrote these results on the board.

Method 1	**Method 2**
309	298
305	310
313	298
311	298
265.5	291
316	49 in^2
310	312.5
312	305.5
305	300
313	300
319	298

"There are more answers the same," Graham said.

"Why do you think so?" I asked.

"I know," Chris blurted out. "It's because we all measured the same-size squares, and that was easier than doing all the counting."

"What's that 49 mean?" Keky asked.

"That one was measured in square inches, not square centimeters," I explained. "You'll be getting your folders back with my comments, and you'll have the chance to make changes. The answer may be sensible, but measuring in square inches makes it difficult to compare with square centimeters."

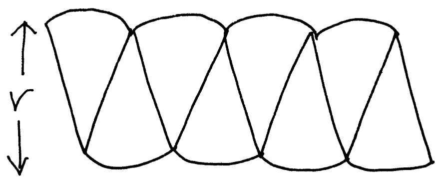
I continued in this way with each method, having students share their reactions and posing questions to have them reflect on the mathematics. As I posted the results for each method, the students broke out into chatter among themselves. They were interested in the information. They were willing to contribute to the class discussion and were animated in doing so.

After discussing their experiences and approximations for the "curvy parallelogram" method, I said to the class, "I'm going to give a short explanation about the curvy parallelogram method that has helped others understand the formula for finding the area inside a circle: πr^2."

When I had their attention, I continued. "You may not understand my explanation," I said. "That's OK, because you'll get a chance to hear it again in math classes, maybe next year and the year after. But see if you can make sense of this now."

I know that teaching by telling doesn't always work and that following someone else's reasoning is difficult. But I decided to take this opportunity to offer an explanation to the students and give them the opportunity to consider the mathematics behind the formula.

I drew the curvy parallelogram on the board. "Instead of measuring the base and height of the parallelogram as you did," I said, "I'm going to think about those measurements differently. First of all, the height of the parallelogram is equal to the radius of the circle. Who can explain how I know that?"



Jason came to the board and drew a radius on one of the sectors to show what I meant.

"The base of the parallelogram," I continued, "is equal to half the circumference. Who can explain that?"

"Because it has four of the eight curvy parts," Meghan said.

"Half is on the top and half is on the bottom," Chris added.

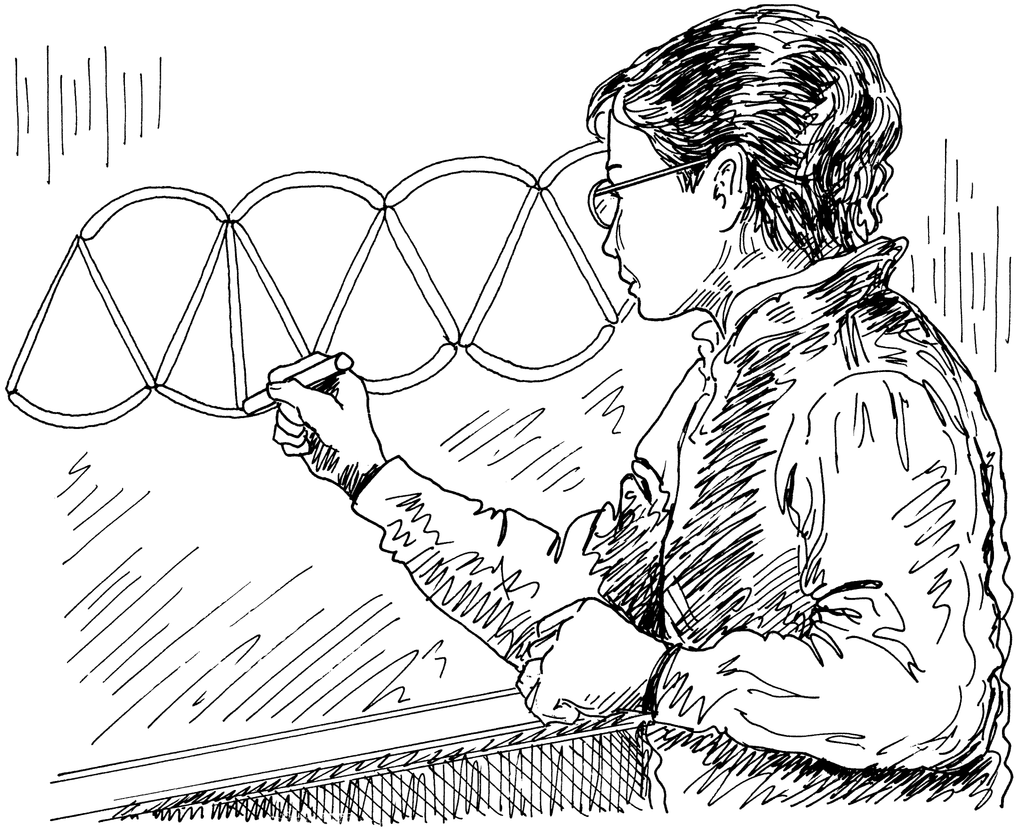
"Remember that the circumference is equal to pi times the diameter," I said. I wrote on the board:

$$c = \pi d.$$

We had investigated that relationship earlier in the year. The students had measured the diameters and circumferences of many circular objects I had brought to class—jar lids of various sizes, a Frisbee, several phonograph records, coasters, paper and plastic plates, plastic cups and bottles, a mirror, a lampshade, a wastepaper basket. The students recorded their measurements on a chart on the chalkboard. I then had the class look at the chart and see what relationships they could find between the diameter and circumference measurements. Calculators were particularly useful for them to verify that the circumference of each circle was approximately 3 times the diameter. In this way, I introduced the idea of pi as a number that was an approximation of 3.

"Since the radius is half the diameter, then half the circumference is pi times the radius," I said. I labeled the base of the parallelogram πr.

"So if I multiply the base times the height," I concluded, "that is πr times r, which is the same as πr^2."



Jason draws a radius on one sector to show it's the height of the curvy parallelogram.

I estimated that about a third of the students followed this explanation. I reassured the class about my purpose in telling them this.

"Don't worry if this doesn't make complete sense," I said. "You'll have other opportunities to learn about this as you study math. If you don't understand this now, you may be able to the next time you encounter it."

Students were interested in seeing their work and my comments. Before I returned their folders, however, I wanted to talk a bit more and give them an additional writing assignment. I was interested to see how their thinking and writing might be different after our class discussion.

"Before I return your work," I said, "I'm interested in whether you would recommend this unit for other classes?"

Their reactions were positive.

"I liked the activities," Amir said, "especially the weighing and using beans. I liked to draw too."

"I think it was interesting to learn different ways to find the area," Lauren said. "I think that would be good for other classes."

"It's a good break from book work," Jeff said.

"It made you think and it was fun," Jason said.

"It was a lot of work," Cory said, "but it kept you involved."

I then said, "I'm also interested in your ideas about what mathematics you think you learned from these activities. For homework, I'd like you to write about what you think these activities helped you learn."

"How much should we write?" Gabe asked.

"At least a page," I said. I find that setting a minimum length provides a useful parameter for students.

An assignment such as this is a useful way to assess what the unit accomplished. Despite our best teaching efforts, the *taught* curriculum and the *experienced* curriculum are not always the same. Having students write is a way to understand their individual perceptions of the learning experience better and to gain insights into their mathematical understandings.

Michael wrote: *This was a good asignment because it expanded our minds to find different ways to find the area of a circle instead of doing the old mathematical way. The ways we did the methods were wierd. I never thought we could do it. But doing this porject is good comunication. We did it with partners so we learned to work together and learn the new methods.*

I learned about that if you shuffle pieces around you could make an easier object to find the area. Like in the curvy parallelagram method we shuffled the pieces around to make an easier object.

Cory wrote: *I really never knew their was so many ways to find out the area of a circle. I learned about circumscribing, inscribing, weighing a circle made out of linoleum, how to find the area with beans, centimeters squared, and how it has to do with a circle, averaging numbers, counting squares and then averaging those numbers together, and experimenting with all these things.*

Math

Circle

What I learned From This

Me and Jeff were partners on an experiment to Find the area of circle, We were given six methods to experiment with.

After we completed the experiment I thought that the experiment really did teach me alot.

One thing was that there is always more than one method in Finding the answer in mathmatics.

That was proven by the six different methods to Find the circles area.

I also learned that finding the area of a circle isn't impossible (even though that is what I thought before).

I learned the methods themselves for finding the area of a circle,

Chris gave his perspective on what he learned from this unit.

What I Learned

I learned a lot of ways to find the area of a circle. I was especially amazed that you could find the area of a circle by weighing. I mean that nobody would think of finding an area by weighing, they would rather think of multipling or something more mathematical.

I learned that you can take something apart and put it back together in a different shape and that could help you find the area of something (like the curvy parellogram method). The same area can be found if you take something apart and put it back together in a different shape.

"What I Learned"

What I learned from this project. was how to work better with people and get along with them better. I also learned how to find the area of a circle in a couple of fun and exciting ways. Such as weighing, counting, etc.. I really don't know when I'll have to use this skill, but I guess its a good skill to have.

Having students write about their experiences is a way to compare the *taught* curriculum and the *perceived* curriculum.

I'm sure that everybody has written this. "I have learned what the area of a circle is which is $\pi \times r^2$." But a teacher just didn't say it when he/she was in the front of the class. They proved *it and thats one of the things that makes this project so special.*

Jason wrote: *What I learned from this project was how to work better with people and get along with them better. I also learned how to find the area of a circle in a couple of fun and exciting ways. Such as weighing, counting, etc. I really don't know when I'll have to use this skill, but I guess its a good skill to have.*

From Lauren: *I learned a lot of ways to find the area of a circle. I was especially amazed that you could find the area of a circle by weighing. I mean that nobody would think of finding an area by weighing, they would rather think of multipling or something more mathematical.*

I learned that you can take something apart and put it back together in a different shape and that could help you find the area of something (like the curvy parellogram method). The same area can be found if you take something apart and put it back together in a different shape.

A Note about a Class of Seventh and Eighth Graders

The unit had a different flavor in a grades seven and eight pre-algebra class. Most of these students knew the formula for finding the area of the circle and quickly figured that the area of the circle was 314 square centimeters. Their focus was to see how close to this answer the approximations were for each of the methods.

I handled the summarizing discussion the same way as I did with the sixth graders. These students' responses, however, reflected their extra year or two of experience with mathematics.

For an assessment at the end of the unit, I gave the students a homework assignment to see how they might apply their understanding of area in another problem situation.

"I'd like you to investigate the prices of different-sized pizzas and see how these prices relate to their areas," I said. I gave them specific directions orally.

"What you're to do," I said, "is to phone a pizza place and find out how big each size pizza is and how much it costs. Then figure whether the prices make mathematical sense with regard to their sizes, whether the prices are proportional to the sizes of the pizzas. If they do make sense, explain why. If not, tell what you would charge and why."

"Does it matter which kind of pizza we ask about?" Jon asked.

"No," I replied, "as long as you use the same kind of pizza for each price."

"Do we have to figure their areas?" Brent asked.

"I don't want to give you hints for this," I said. "I'm interested in how you'll go about solving this without my help."

"Could you say the problem again?" Jenny asked.

I repeated the problem.

"Do you have it written down?" Mike said. The class was accustomed to getting their assignments in writing.

"No, I haven't written it down," I said. "Would you each take out a sheet of paper now and write the problem down in your own words?"

In this way, I took the opportunity to give them experience with formulating the problem in their own ways. I gave them time to do this and then had several students read what they had written.

Amanda wrote: *Call up a pizza place and ask the size of each of their pizzas—small, medium, and large. Then ask the prices. See if the prices are proportional to the areas of the pizzas. If they are not, reprice them.*

Brent wrote: *For this problem we are supposed to determine whether the price of pizza is proportional to the different sizes you can order.*

Allison wrote: *Call or visit a pizza place. Find out the size and price of pizzas. Then decide whether it's mathematically sensible or not. How much would you charge?*

Jennifer wrote: *What we are supposed to do is to call or visit a pizza place and find out what the price of each pizza is and what size it is. We then have to decide whether or not the prices are acceptable. If they aren't, what would we change them to?*

Geoff wrote: *What are the prices of a small, medium, and large pizza? Do they make mathematical sense? If so, why? If not, what should they charge?*

Most of the students figured the area of each pizza in square inches and then figured how much a square inch of pizza costs for each size. In most cases, the larger pizzas were less expensive per square inch. Some students decided the prices were close enough and were fine. Others presented alternative pricing. Their solutions and explanations gave me information about what they understood, not only about the area of circles, but about other mathematical ideas as well.

Final Thoughts

I saw a sign while traveling in Colorado that read: "Hindsight is always 20-20 vision." In that light, an idea occurred to me about another way to assess students' understanding about area. After completing this unit, I'd like students to consider an irregular bloblike shape and decide which of the methods they used to approximate the area of the circle would be appropriate for the blob. I'd ask them to explain why the ones they identified made sense and why the others wouldn't. I'd also ask them to choose the method they think would be "best" for approximating the area and to explain why.

I'm interested in starting the unit another time by having students first try their own ways to approximate the area of a circle before I introduce them to the methods suggested in the *Arithmetic Teacher* article. I'd like students to understand that mathematical ideas and methods are invented by people engaged in looking for relationships and connections.

I was pleased with the unit and look forward to presenting it again to students. Along with contributing to students' understanding about finding the area of circles, the unit encouraged students to communicate about mathematical ideas. It gave them experience with using data to judge the effectiveness of mathematical methods. Also, it gave them a mathematical experience that involved them with several areas of the curriculum: number, geometry, measurement, and logical reasoning.