Beyond Invert and Multiply: Make Sense of Fraction Computation

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CMC - North
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8th Grade, NAEP 1996

Estimate the answer to $\frac{12}{13} + \frac{7}{8}$

Answer choices were:

A.	1	7%
B.	2	24%
C.	19	28%
D.	21	27%
E.	DK/No answer	14%

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"The difficulty with fractions (including decimals and percents) is pervasive and is a major obstacle to further progress in mathematics. . ."

—*Report of the National Math Panel*, March 2008

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4th Grade & 8th Grade NAEP, 2003

Jim has $\frac{3}{4}$ of a yard of string which he wishes to divide into pieces, each $\frac{1}{8}$ of a yard long. How many pieces will he have?

4th	8th	4th	8th
19%	15%	27%	54%
30%	21%	22%	8%

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8th Grade, NAEP 2007

Add the numbers $\frac{7}{10}$, $\frac{7}{100}$, and $\frac{7}{1000}$.

Write this sum as a decimal.

Correct:	46%
Incorrect:	51%
Omitted:	3%

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Why are fractions so hard?

- Fraction notation – numbers must be considered in new ways
- Practices that simplify and/or mask the meaning of fractions
- Many meanings of fractions
- Overreliance on whole number knowledge

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Brendan, Grade 4

1. Without computing the exact answer, decide which of these expressions would produce the answer with the least value and the greatest value.

A. Addition: $\frac{3}{4} + \frac{5}{8}$

B. Subtraction: $\frac{3}{4} - \frac{5}{8}$

C. Multiplication: $\frac{3}{4} \times \frac{5}{8}$

D. Division: $\frac{3}{4} \div \frac{5}{8}$

Least Value Division

Greatest Value multiplication

Explain your thinking below:

Division because you have tol split it up.
Multiplication because you have to add more than 2 of it.

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"Students make mistakes not because they are ***not*** thinking, but because they ***are*** thinking."

-Constance Kamii

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What is Fraction Sense?

Fraction sense is tied to common sense: Students with fraction sense can reason about fractions and don't apply rules and procedures blindly - nor do they give nonsensical answers to problems involving fractions.

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Common Core State Standards

Number and Operations – 3.NF

Develop understanding of fractions as numbers.

1. Understand a fraction 1/*b* as the quantity formed by 1 part when a whole is partitioned into *b* equal parts; understand a fraction *a*/*b* as the quantity formed by *a* parts of size 1/*b*.
2. Understand a fraction as a number on the number line; represent fractions on a number line diagram.
3. Explain equivalence of fraction in special cases, and compare fractions by reasoning about their size.

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"It may be surprising that, for most students, to think of a rational number as a number – as an individual entity or a single point on a number line – is a novel idea."

—*Adding it Up: Helping Children Learn Mathematics*, National Research Council, 2001

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Common Core State Standards

Number and Operations – 4.NF

Extend understanding of fraction equivalence and ordering.

Build fractions from unit fractions by applying and extending previous understandings of operations on whole numbers.

Understand decimal notation for fractions, and compare decimal fractions.

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Common Core State Standards
Number and Operations – 5.NF

Use equivalent fractions as a strategy to add and subtract fractions.

Apply and extend previous understandings of multiplication and division to multiply and divide fractions.

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Common Core State Standards
Number and Operations – 6.NS

Apply and extend previous understandings of multiplication and division to divide fractions by fractions.

Apply and extend previous understandings of numbers to the system of rational numbers.

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Common Core State Standards
Number and Operations – 7.NS

Apply and extend previous understandings of operations with fractions to add, subtract, multiply, and divide rational numbers.

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Common Core State Standards

Number and Operations – 8.NS

Know that there are numbers that are not rational, and approximate them by rational numbers.

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Common Core Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

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Understanding Fraction Equivalency through Measurement

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Common Core State Standards
Number and Operations – Fractions

Grade 3: Explain **equivalence** of fraction in special cases, and compare fractions by reasoning about their size.

Grade 4: Extend understanding of fraction **equivalence** and ordering.

Grade 5: Use **equivalent** fractions as a strategy to add and subtract fractions.

Grades 6-8: Apply and extend previous understandings…

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Why Measurement?

- Fractions occur naturally in measurement activities
- Presents fractions a unit of measure
- Highlights the importance of the unit
- Supports connections between a fraction and a point on a number line
- Can be applied to any fraction representation

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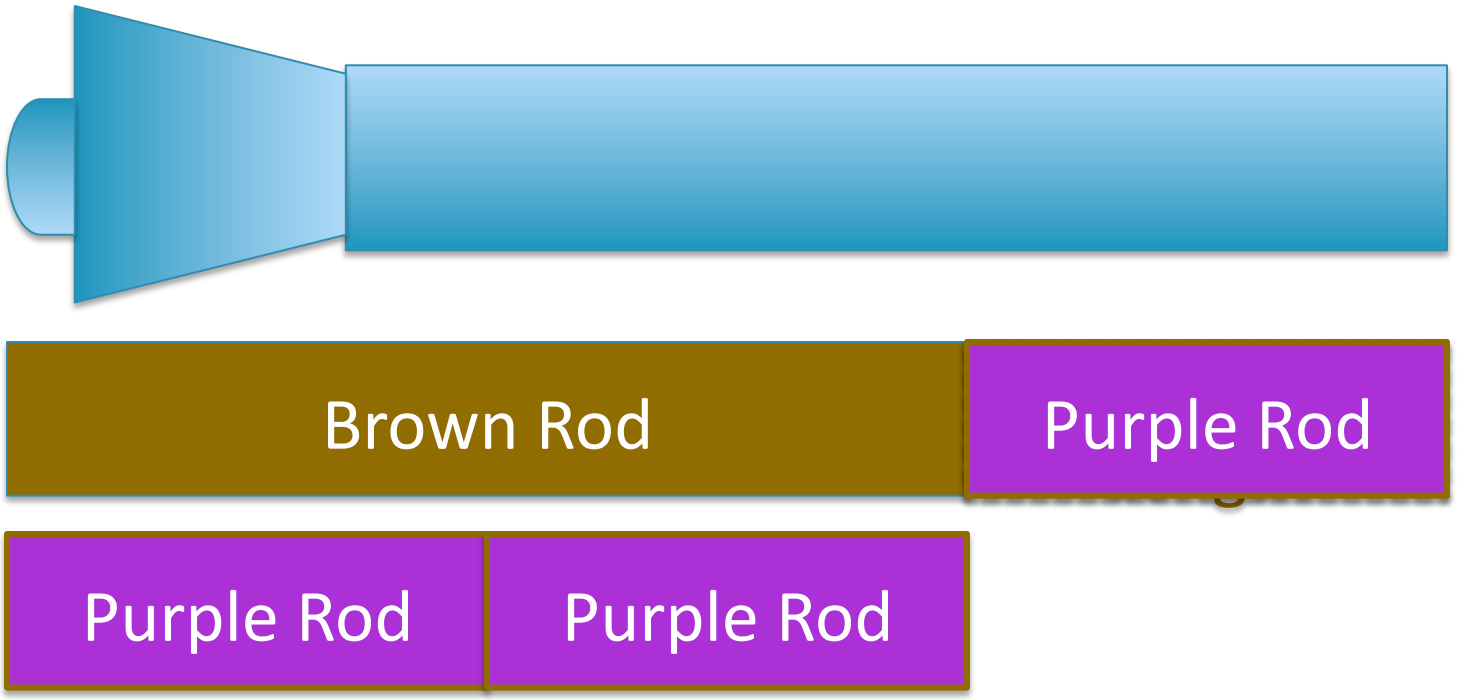
Measure the marker with brown rods



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Common Core State Standards

Number and Operations – 3.NF

Understand a fraction 1/*b* as the quantity formed by 1 part when a whole is partitioned into *b* equal parts; understand a fraction *a*/*b* as the quantity formed by *a* parts of size 1/*b*.

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Before we begin – a comment about Cuisenaire Rods

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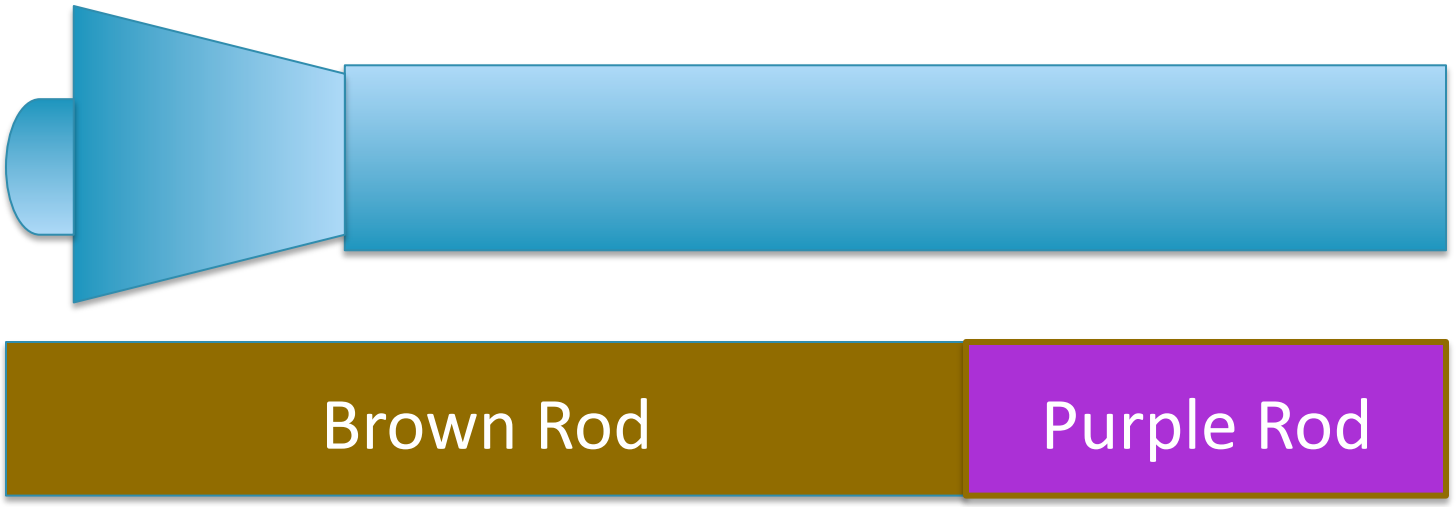
The marker = $1\frac{1}{2}$ brown rods long



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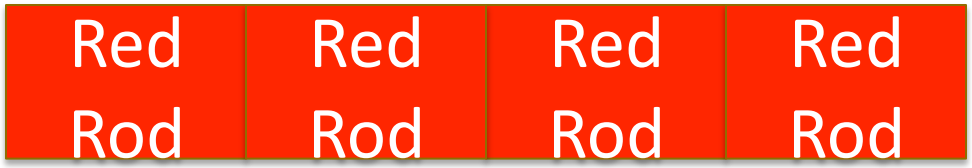

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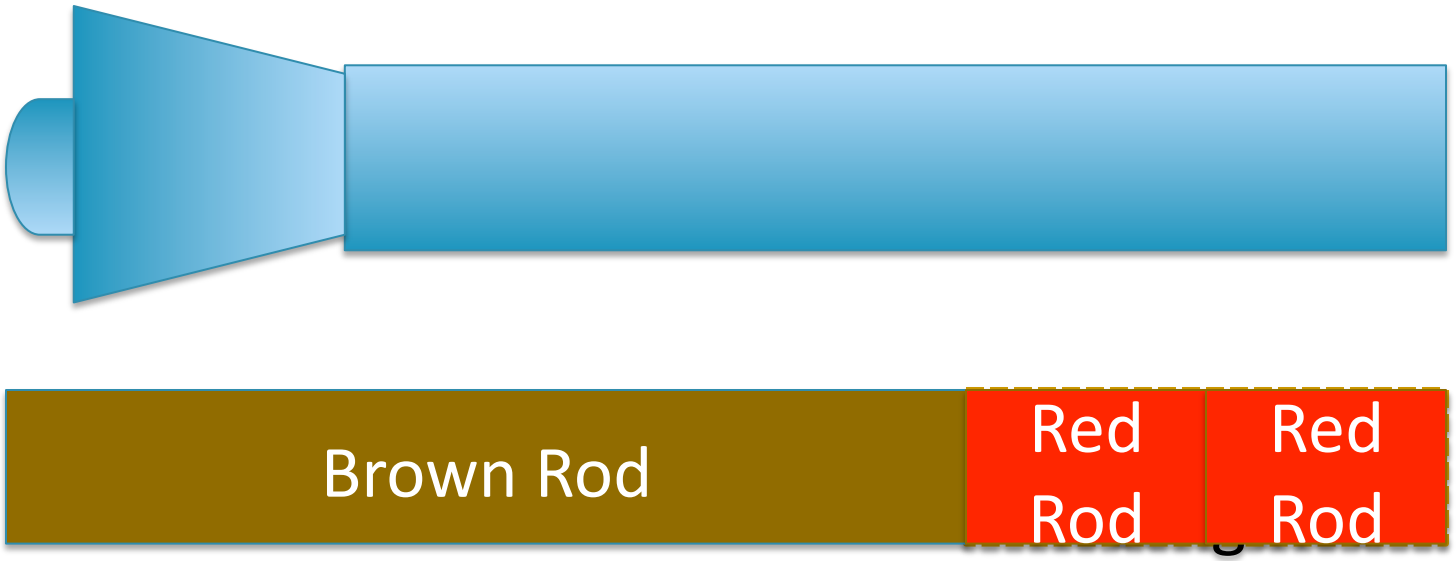
The marker = $1\frac{2}{4}$ brown rods



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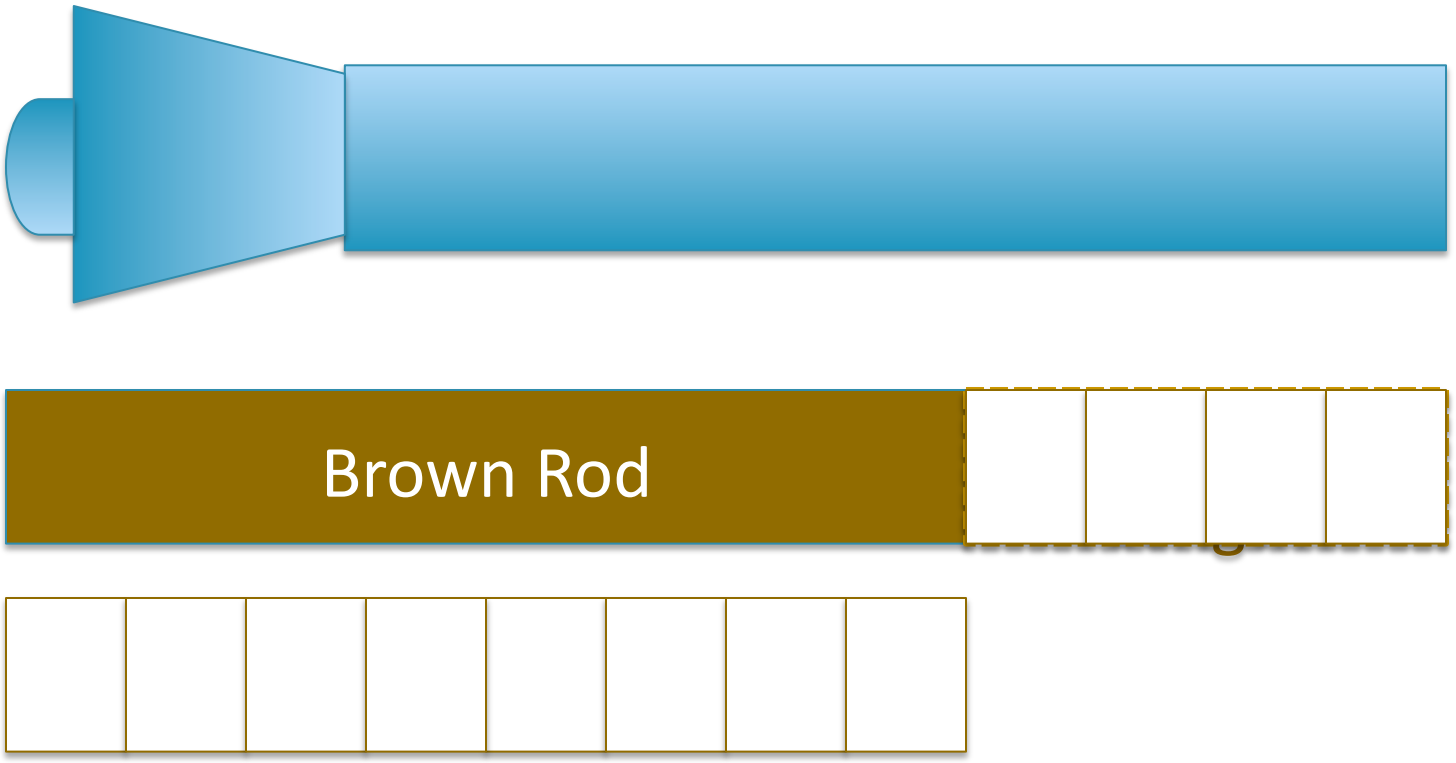
Measure the marker with brown rods



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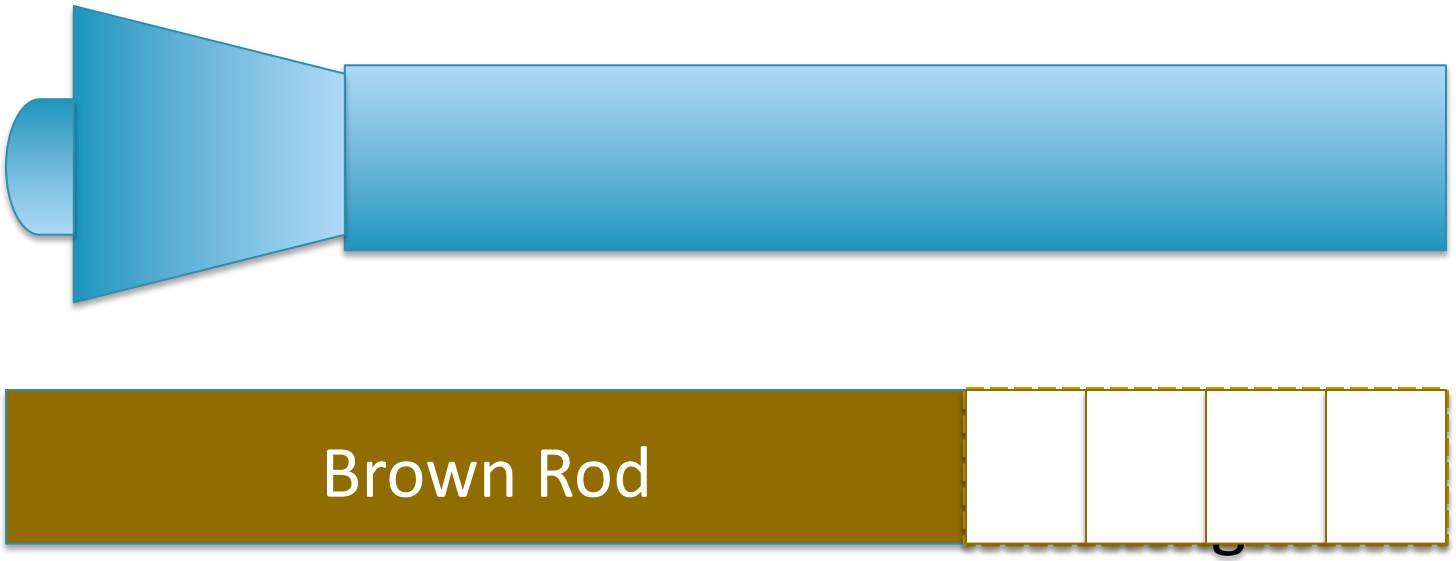
The marker = $1\frac{4}{8}$ brown rods



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Item	First Way	Second Way	Third Way
Marker	1 ½ br. rods	1 2/4 br. rods	1 4/8 br. rods
Pencil			
Book			

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"The context in which any vehicle – concrete or pictorial – is used is as important as the material itself. By context, I mean the ways in which students work with the material, toward what purposes, with what kinds of talk and interaction."

-Deborah Ball, Dean of the School of Education, University of Michigan

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"The context in which any vehicle – concrete or pictorial – is used is as important as the material itself. By context, I mean the ways in which students work with the material, **toward what purposes**, with what kinds of **talk and interaction**."

-Deborah Ball, Dean of the School of Education, University of Michigan

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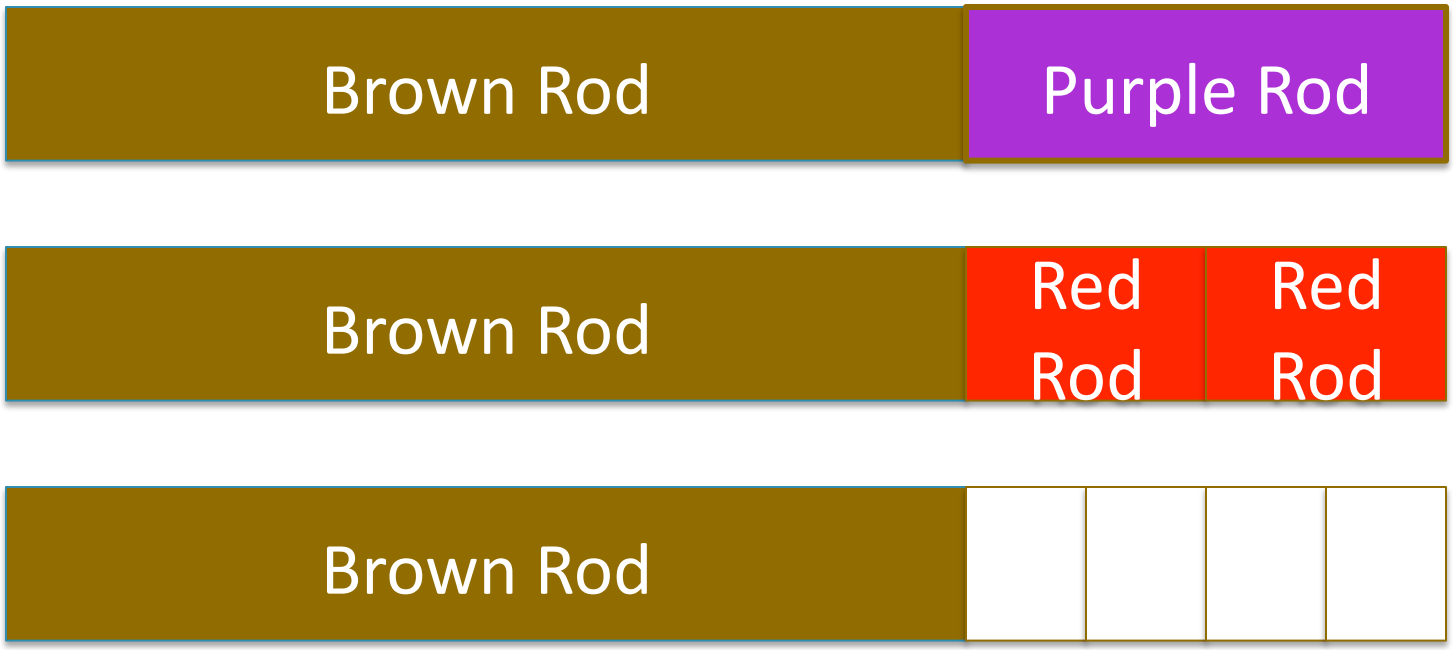
$$1\frac{1}{2} = 1\frac{2}{4} = 1\frac{4}{8}$$



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Fraction Sense and Operations with Fractions

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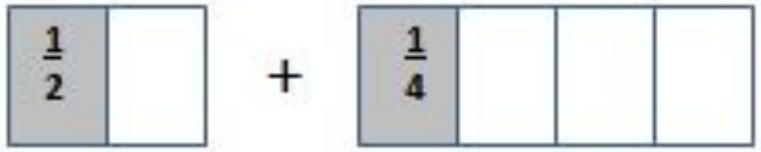
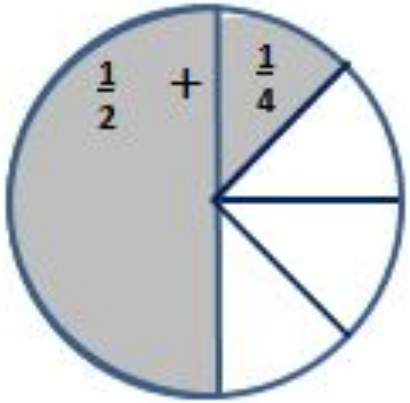
Which of the following pictures could be used to correctly solve $\frac{1}{2} + \frac{1}{4}$?

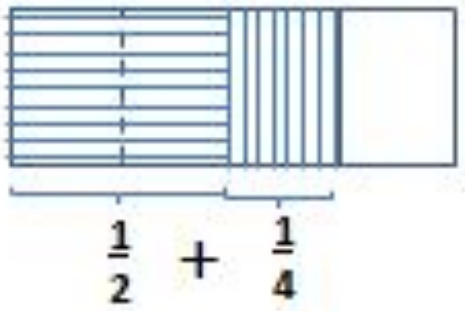
A.



B.



C.



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Find the sum

$$\frac{2}{3} + \frac{1}{4} = \frac{3}{7}$$

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Find the sum

$$\frac{2}{3} + \frac{1}{4} = \frac{3}{7}$$

Is the answer greater or less than $\frac{1}{2}$?

Is the answer greater or less than 1?

What else do I know about the answer?

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Adding on the Number Line

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Number Line Principles

(adapted from the Learning Mathematics through Representations Project at UC Berkeley)

- Numbers increase in value from left to right. Numbers decrease in value from right to left.
- The value of a number is based on its distance from zero.
- The "unit interval" is the distance from 0 to 1 (or any equivalent distance on the line).
- As long as any 2 numbers are shown on the line the unit interval can be determined.

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Number Line Principles

(adapted from the Learning Mathematics through Representations Project at UC Berkeley)

- The unit interval can be divided further into sub-units.
- The denominator of a fraction tells how many sub-units the unit interval has been divided into.
- Every number has a place on the number line, but not every number needs to be shown.

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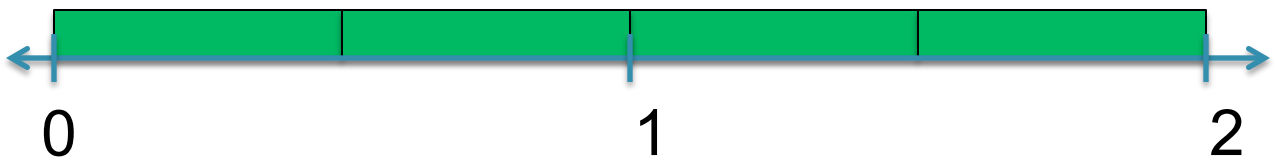
Making a Number Line

- Take out 4 dark green rods.
- Use 2 dark green rods to create your unit interval. → One dark green rod equals ½ of the unit interval.



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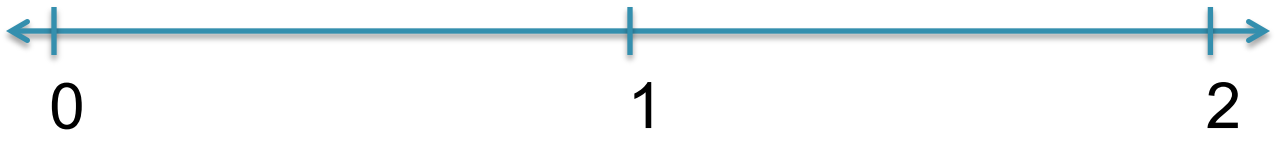
2-Unit Number Line



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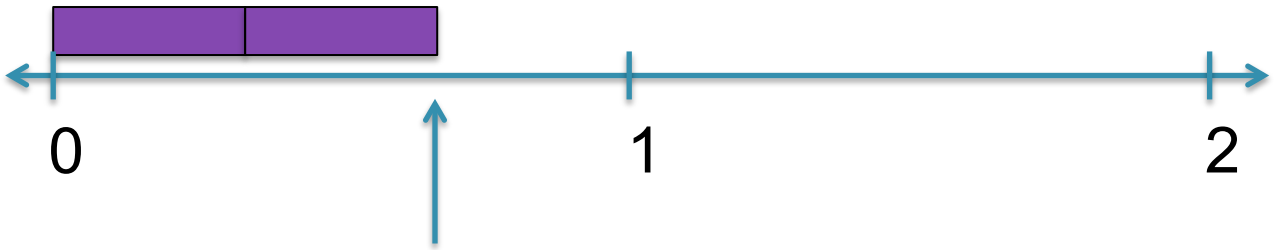
Find the rod that equals $\frac{1}{3}$ of the unit interval

- Use the rods to solve $\frac{1}{3} + \frac{1}{3}$



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To solve $\frac{1}{3} + \frac{1}{3}$:

- Did you need to find any equivalent fractions?
- Did this idea come up?

➢ *Understand a fraction 1/b as the quantity formed by 1 part when a whole is partitioned into b equal parts; understand a fraction a/b as the quantity formed by a parts of size 1/b.*

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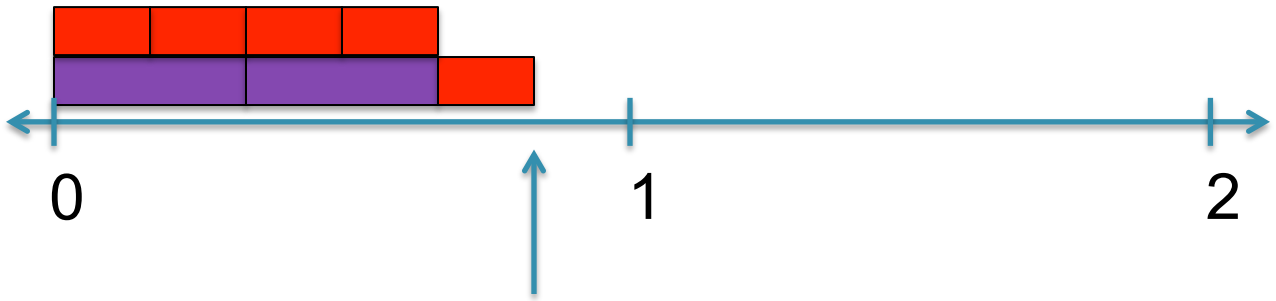
Find the rod that equals $\frac{1}{6}$ of the unit interval

- Use the rods to solve $\frac{2}{3} + \frac{1}{6}$



$$\frac{2}{3} = \frac{4}{6}$$

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To solve $\frac{2}{3} + \frac{1}{6}$:

- Did you need to find any equivalent fractions?
- Did this idea come up?

➢ *Understand a fraction 1/b as the quantity formed by 1 part when a whole is partitioned into b equal parts; understand a fraction a/b as the quantity formed by a parts of size 1/b.*

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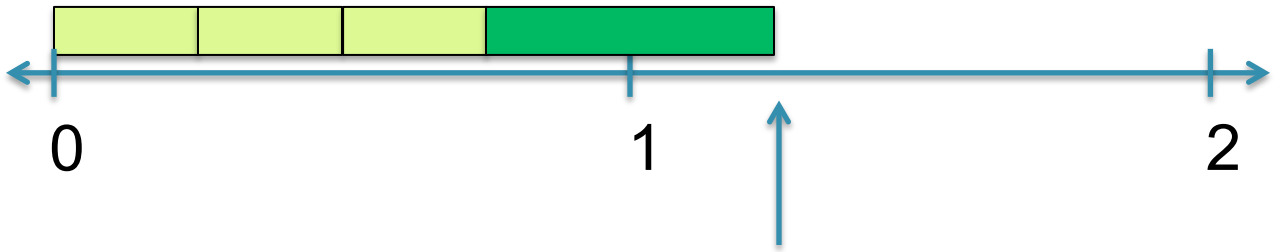
Find the rod that equals $\frac{1}{4}$ of the unit interval

- Use the rods to solve $\frac{3}{4} + \frac{1}{2}$



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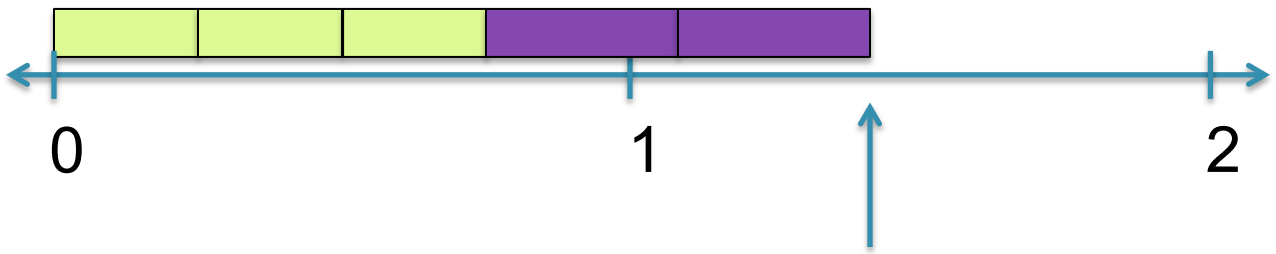
Use the rods that equal $\frac{1}{4}$ and $\frac{1}{3}$ of the unit interval

- Solve $\frac{3}{4} + \frac{2}{3}$



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Use the rods to solve the following problems:

- $\frac{3}{4} + \frac{5}{6}$
- $\frac{3}{2} + \frac{1}{6}$
- $\frac{1}{3} + \frac{1}{2}$
- $\frac{1}{4} + \frac{2}{3}$
- $\frac{2}{3} + \frac{5}{12}$
- $\frac{11}{12} + \frac{1}{2}$

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Moving beyond the rods…

- How can this experience help students think about adding without using the rods?
- What important ideas are students drawing upon?
- What important ideas are students developing?

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Common Core State Standards
Number and Operations – Fractions

Grade 5: Use equivalent fractions as a strategy to add and subtract fractions.

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Find the sum

$\frac{4}{6}$ $\frac{6}{9}$ $\frac{12}{18}$

$\frac{8}{12}$ $\frac{10}{15}$

$$\frac{2}{3}+\frac{1}{4}=\frac{11}{12}$$

$\frac{2}{8}$ $\frac{4}{16}$

$\frac{3}{12}$

1. Start with finding equivalent fractions.

2. Then find common denominators..

3. Add, and check to see if my answer makes sense.

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Finally.....

- $a \times 1 = a$

$$\frac{b}{b} = 1$$

$$a \times \frac{b}{b} = a$$

Multiplicative identity property of 1

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A student does the following multiplication problem:

$$\frac{5}{6} \times \frac{2}{2} = \frac{10}{12}$$

Look at the statement below:

$\frac{10}{12}$ is twice as large as $\frac{5}{6}$.

60% of 4th graders and 51% of 6th graders agreed with the statement.

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A student does the following division problem:

$$\frac{6}{10} \div \frac{2}{2} = \frac{3}{5}$$

Look at the statement below:

$\frac{3}{5}$ is half the size of $\frac{6}{10}$.

73% of 4th graders and 57% of 6th graders agreed with the statement.

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Fraction Sense and Multiplication with the Number Line

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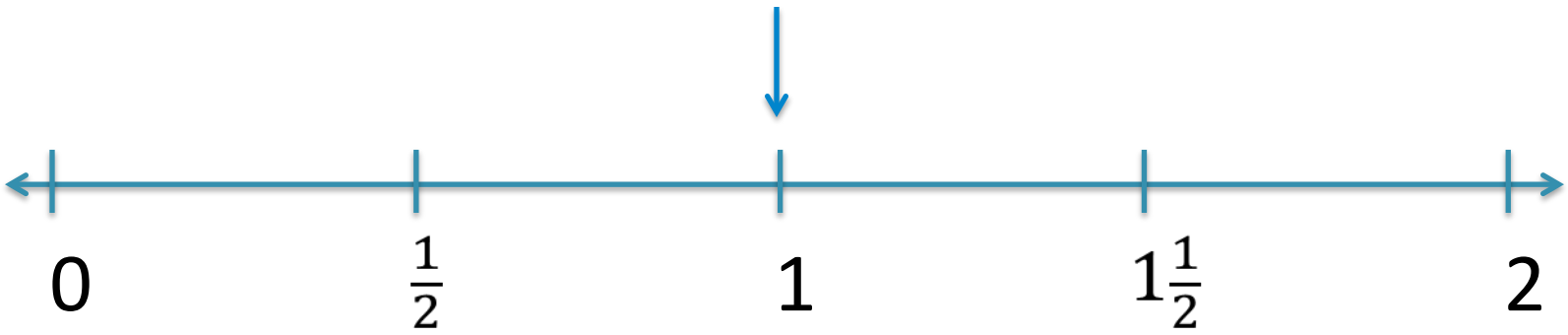
$$2 \times \frac{1}{2}$$

- $2 \text{ of } \frac{1}{2}$ or $\frac{1}{2} \text{ of } 2$



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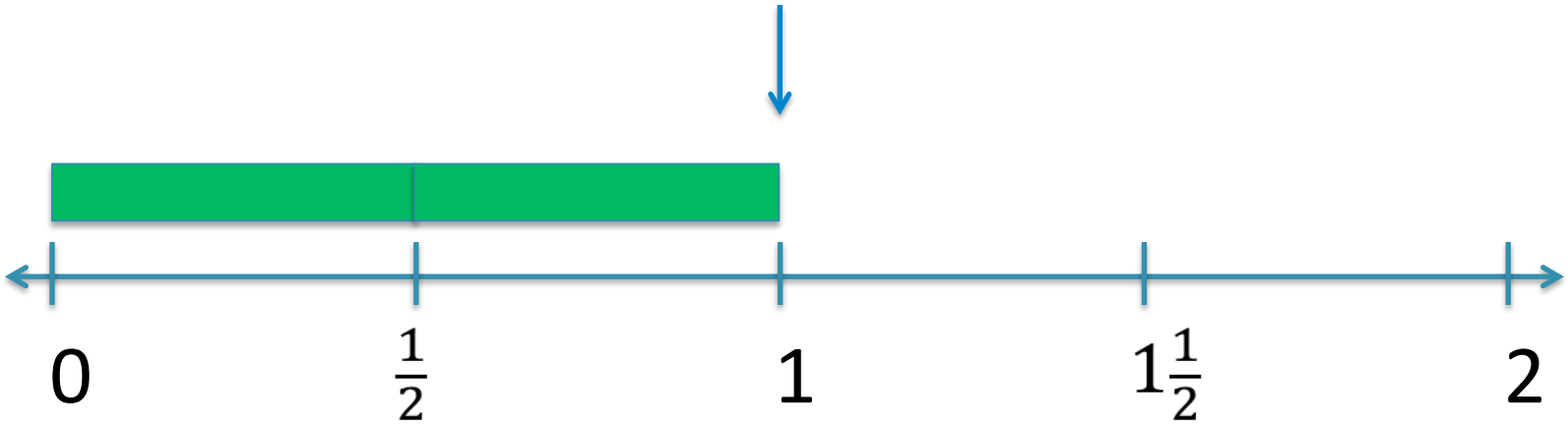
$$2 \; of \; \frac{1}{2}$$



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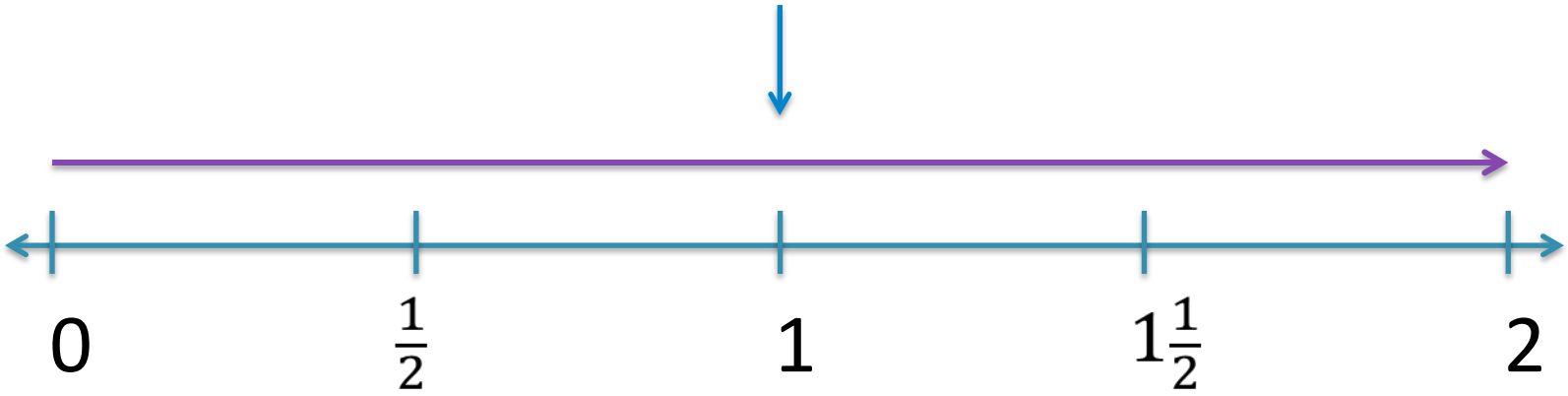
$$\frac{1}{2} \text{ of } 2$$



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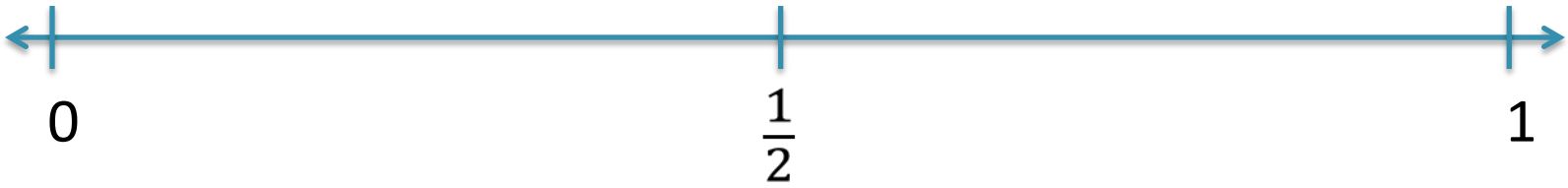
$$\frac{5}{6} \times \frac{1}{2}$$

- $\frac{5}{6}\ of\ \frac{1}{2}$ or $\frac{1}{2}\ of\ \frac{5}{6}$



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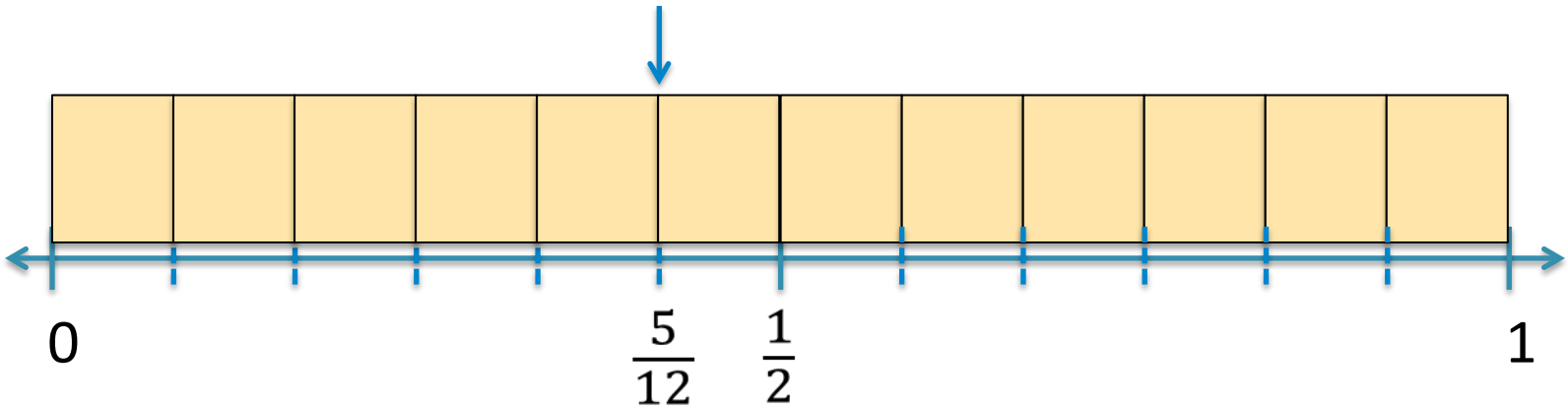
$$\frac{5}{6} of \frac{1}{2}$$



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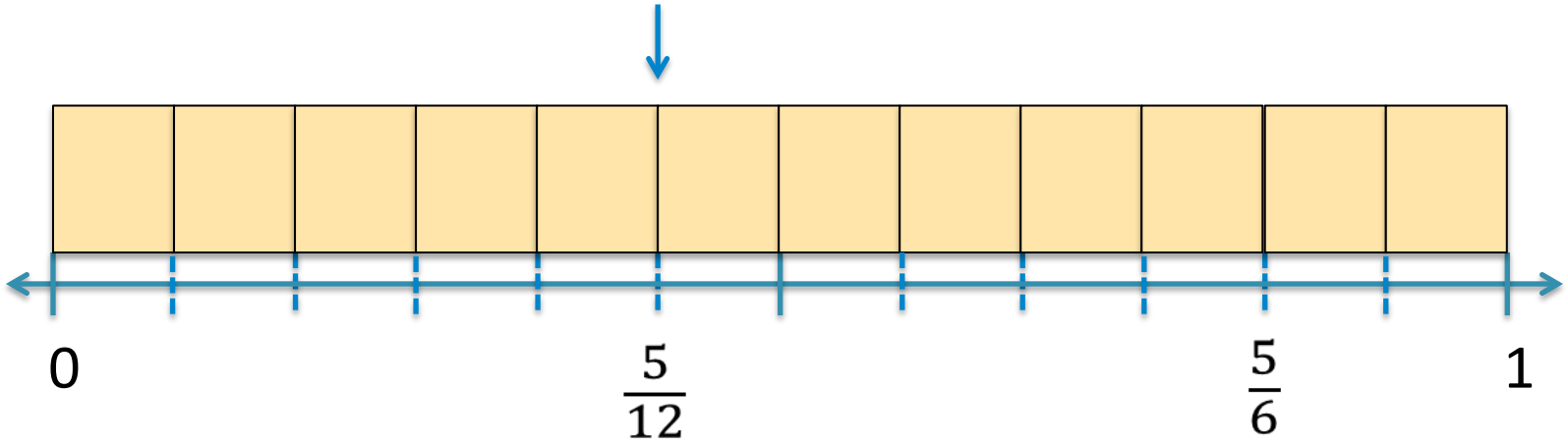
$$\frac{1}{2} of \frac{5}{6}$$



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Fraction Sense and Division

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Yours is not to reason why;
just invert and multiply.

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Consider…..

- $6 \div 2$: How many 2s are in 6?
- $6 \div 1$: How many 1s are in 6?
- $6 \div \frac{1}{2}$: How many $\frac{1}{2}$s are in 6?
- $6 \div \frac{2}{3}$: How many $\frac{2}{3}$s are in 6?
- $\frac{2}{3} \div \frac{1}{2}$: How many $\frac{1}{2}$s are in $\frac{2}{3}$?

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NAEP, 2003 (Modified)

- Jim has $\frac{5}{6}$ of a yard of string which he wishes to divide into pieces, each $\frac{1}{12}$ of a yard long. How many pieces will he have?

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Something else to consider…

- You have 28 students in your class.
- You want to split them into 2 equal groups.
  - Use words to describe how you did this.
  - Write an equation to describe how you did this.

$$28 \div 2 = 14$$

$$28 \times \frac{1}{2} = 14$$

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Inverse Operations

Addition

A number's *additive inverse* is the number that you can add to the original number to get the additive identity (0). For example, the additive inverse of 67 is -67, because 67 + -67 = 0, the additive identity.

- *Adding a number is the same as subtracting its inverse.*

$$6 + {}^{-}3 = 6 - 3$$

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Inverse Operations

- **Multiplication**

A number's *multiplicative inverse* is the number that you multiply the original number by to get the multiplicative identity (1). Since $6 \times \frac{1}{6} = 1$ the multiplicative inverse of 6 is $\frac{1}{6}$.

- *Dividing by a number is the same as multiplying by its inverse.*

$$6 \div 2 = 6 \times \frac{1}{2}$$

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Invert and Multiply

$$\frac{3}{2} \times \frac{4}{15}$$

In order to understand this….

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Students need to understand this…

- $24 \div 6$

$2 \times 24 \div 2 \times 6$

$\frac{1}{2} \times 24 \div \frac{1}{2} \times 6$

Multiplying or dividing the dividend and divisor by the same number will not change the quotient.

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In addition – you really can divide across numerators and denominators…

- $\frac{4}{15} \div \frac{2}{3}$

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Sometimes it's more elegant than others….

- $\frac{4}{15} \div \frac{3}{2}$

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Sometimes, well not so much…

$$\frac{4}{15} \div \frac{3}{2} = \frac{\frac{4}{3}}{\frac{15}{2}}$$

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Another Proof

$$\frac{4}{15} \div \frac{3}{2} =$$

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What is Fraction Sense?

Fraction sense is tied to common sense: Students with fraction sense can reason about fractions and don't apply rules and procedures blindly - nor do they give nonsensical answers to problems involving fractions.

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Common Core Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

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What can you do to help?

- Support students' understanding of fractions as numbers.
- Support students' understanding of equivalence.
- Focus on properties of arithmetic.
- Help students appropriately connect whole-number knowledge to working with fractions.
- Provide opportunities for students to reason about fractions.

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"Students make mistakes not because they are ***not*** thinking, but because they ***are*** thinking."

-Constance Kamii

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"Children who are successful at making sense of mathematics are those who believe that mathematics makes sense."

-Lauren Resnick

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Thank you!!

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