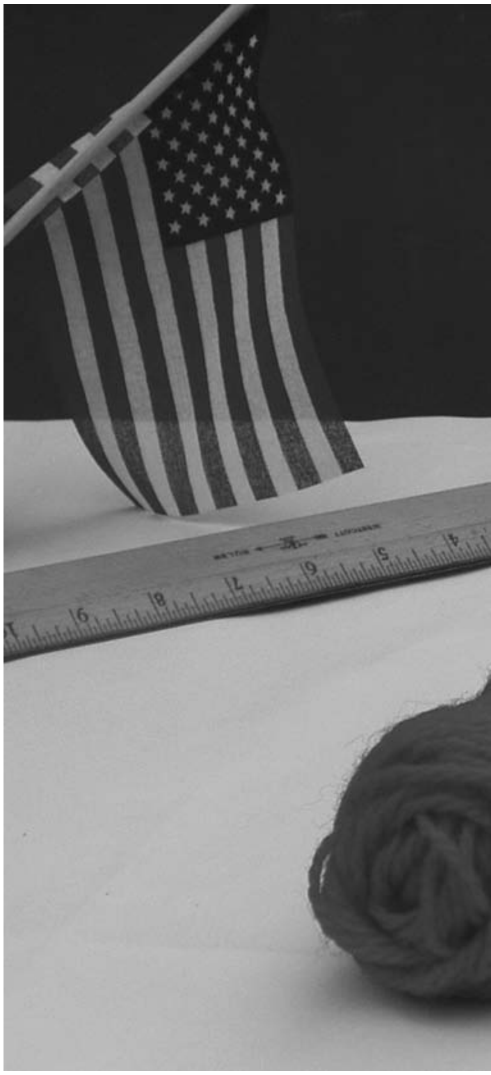

A can of Coke leads to a piece of pi

A professional development exercise for educators is an adaptable math lesson for many grades

BY MARILYN BURNS

During a professional development session for K-8 teachers, I held up a can of Coke and a foot-long piece of yarn. Wrapping the yarn once around the can, I asked the group, "How will the length of yarn wrapped once around the can compare to the can's height? Will it be taller, shorter, or about the same length as the height of the can?"

The teachers thought about the puzzle for a few minutes, then I asked for a show of hands for each of the three possibilities. The group split pretty evenly among the choices. I unwrapped the yarn, marking with my finger the length showing the can's circumference, and held this section up to the can. Teachers gasped. Many were surprised to see that the section of yarn was almost twice as long as the can was tall.

I repeated the procedure with several other containers I had collected — a plastic water bottle, coffee mug, plastic drinking cup, a Styrofoam coffee cup. Each time, I wrapped the yarn around the container, asked for predictions, and then compared the yarn's length to the container's height. I asked teachers to predict each time, and they did so correctly for some containers and incorrectly for others. Some teachers, however, chose not to make predictions, preferring instead to observe the results from the additional trials. All seemed curious.

Rather than calling attention to correct or incorrect responses, or to teachers who chose not to make predictions, I tried to maintain an atmosphere of curiosity and playfulness,

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PHOTO BY BOB ROSSBACH

two qualities often missing from math lessons. My goal was to provide a professional development experience that not only would help teachers learn more about the mathematics they have to teach, but also would help them learn what's needed to teach it. I wanted to focus on three issues — the mathematics they have to teach, how children learn mathematics, and the elements that are important for effective classroom instructional strategies.

MATHEMATICS AND SENSE MAKING

Essential to all professional development in mathematics is the idea that making sense of mathematics is key to learning. Just as learning to read calls for bringing meaning to the printed page, learning math calls for bringing meaning to mathematical symbols, concepts, and skills.

For many, however, learning math was never grounded in sense making. Elementary school math instruction focuses on how to carry, borrow, follow the steps for multiplying or doing long division, work with fractions, and so on. From these approaches, many students perform successfully on paper-and-pencil assignments. ("Yours is not to question why, just invert and multiply" can produce this result.) But a serious risk of focusing on procedures is that while many students can perform them correctly, they do not understand why they work. Also, they see learning mathematics as learning a collection of procedures to be followed. Even worse, they do not expect mathematics to make sense.

We tend to teach the way we were taught, and, without professional development, teachers who were taught mathematics with a focus on learning and practicing procedures are likely to teach in their own classrooms with the same focus. Also, many teachers, especially those teaching the lower elementary grades, feel insecure about their own mathematical knowledge. Their lack of a firm base of mathematical knowledge compounds the problem — teachers cannot teach what they don't understand.

For many, learning math was never grounded in sense making.

Back to the Coke can. How can this experience contribute to teachers' awareness of the value of making sense of math and also to their understanding of math? To respond to that question, we need to think about how to predict for any container whether the wrapped-around yarn is taller, shorter, or the same as the container's height. What's the math involved? Is there a way to make sense of this situation or is it more of a mathematical magic trick?

The Coke can experience is indeed a mathematical problem, and one that references what we were taught in the elementary grades about circles. When we were introduced to circles in school, we all learned about pi. When asked what they remember about pi, teachers typically offer that pi is worth $3^1/_7$ or 3.14. Some dredge up the standard formulas, or versions of them: $A = \pi r^2$, $c = \pi d$, and $c = 2\pi r$. Typically someone offers the information that the values they learned for pi aren't exact, that the decimal representation includes many more digits and that $3^1/_7$ is just an approximation. Where do these approximations for pi come from? Why do the formulas make sense? Why does the "real" pi have so many more digits in its decimal representation? Teachers rarely are able to explain.

Without formal measuring, I can see that a piece of yarn equal in length to three of my pinky fingers will be much longer than the height of the can.

So, what mathematics can be learned from the Coke can experience? The mathematics has to do with understanding pi not merely procedurally but in a way that goes beyond being able to plug it into formulas. Here's the skinny: Pi is a relationship. If you measure the circumference of any circle, and divide that number by the length of the circle's diameter, you get an answer that's pretty close to 3.14. This holds true for every circle, no matter its size. Or, if you multiply the diameter of a circle by 3.14, the result gives you a close approximation to the measurement of the circle's circumference. Pi is the name given to the ratio between the circumference and diameter of all circles.

If you trace the bottom of a Coke can onto a piece of paper, you draw a circle with a diameter a little longer than my pinky finger. Because I understand the relationship between the circumference and diameter of circles, I know that the distance around the can is a little more than three times as long as my pinky. Without formal measuring, I can see that a piece of yarn equal in length to three of my pinky fingers will be much longer than the height of the can. By using my understanding of the relationship between a circle's diameter and circumference, I can generally predict correctly whether yarn that goes around a container will be longer, shorter, or about the same length as the container's height. If, however, I only learned about pi procedurally, I might be able to plug pi into formulas and arrive at correct answers on exercises, but I wouldn't have the understanding to make sense of the Coke can problem.

Can you imagine what a container would look like if the yarn around the

Math Solutions in Buffalo Public Schools
Buffalo, New York

Since fall 2000, Buffalo Public Schools, an urban district that serves nearly 43,000 students, has used Math Solutions in a focus on improving math instruction. Professional development has included five-day courses over four summers and more than 40 follow-up sessions for teachers, teacher leaders, and administrators.

The district has seen positive effects on student achievement. The 2000-2001 New York State Math Assessment results for 4th and 8th grades in schools where teachers had participated in Math Solutions professional development (schools defined as "greatest need") were compared with schools where teachers had not participated.

- On average, the Math Solutions participating schools showed a 12% increase — double that of the schools that didn't participate in Math Solutions.
- While scores in 8th grade dropped slightly districtwide, they dropped less than 1% at the Math Solutions participating schools, vs. a 4% drop in the schools that didn't participate in Math Solutions.
- More recently, Buffalo's results in 8th-grade math (2000-2001 to 2001-2002) increased 9.3%. This is the highest increase in 8th-grade scores among the Big Five school districts in New York, which include Buffalo, Rochester, Syracuse, Yonkers, and New York City.
- Further, the percentage of 8th-grade students scoring at Levels 3 and 4, the two highest levels, increased from 16% to 25.3% between these years.

The positive response to the professional development from classroom teachers, teacher leaders, and the administrators who observed them, coupled with the test score data, led the school board, despite severe budget cuts, to support continued professional development in Math Solutions, helping leaders deepen their understanding of math content.

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container were just about as long as the height of the container? Think of a can of three tennis balls. That container is about as tall as the diameters of the three tennis balls, very close to pi times the diameter, which is how we figure the circumference of a circle when we know the diameter. (Remember $c = \pi d$?)

Experiences such as this one with the Coke can are needed to help teachers think about the mathematics in all areas of the curriculum. The area of number and operations, which receives the most emphasis in the elementary grades, is especially important to address. Teachers must understand why we "bring down" in long division, why multiplying across the tops and bottoms produces the correct answer when multiplying fractions, why inverting and multiplying works for dividing fractions, and so on. Procedures are important, but how we teach procedures and what we expect students to learn are also important. In addition, teachers need to think about numbers more flexibly, in ways that support the development of number sense. For example, it's clear that 8 + 4 is equal to 7 + 5. But why is this so? And why doesn't it hold for multiplication; that is, why isn't 8 x 4 equal to 7 x 5? Why, when you add two odd numbers, is the answer always even, but when you multiply two odd numbers, the answer is always odd? Teachers benefit from thinking about ideas we typically think of as elementary mathematics. Only when teachers' understanding is robust can they begin to think about how to make these ideas accessible to their students.

To this end, teachers benefit from professional development experiences that involve two components. One is to engage them in thinking about numerical relationships. In a professional development setting, this typically involves presenting a question or investigation, asking teachers to think

Can you imagine what a container would look like if the yarn around the container were just about as long as the height of the container? Think of a can of three tennis balls.

about it on their own first, and then asking them to talk about their ideas in small groups with the goal of finding a way to explain to the larger group what is happening and why.

Here's an example I did recently with a group of teachers. I wrote a simple division problem on the board: 6 ÷ 2 = 3. Then I asked, "What happens to the answer, the quotient, if I double both the dividend and the divisor?" After a moment, everyone agreed that the answer remains the same: 12 ÷ 4 = 3. Then I presented the investigation, "Does this work all the time, and, if so, why?" I asked the teachers to think about this by themselves first and then had them talk in their groups about what they thought.

One of my mathematical goals was to discuss different ways to think about division. For example, 6 ÷ 2 can be thought of as "dividing six into groups of two," and also as "dividing six into two groups." In the first instance, the answer of three refers to the number of groups that result; in the second, it refers to how many are in each group. Either interpretation can be used to help explain. Also, I wanted to relate division to fractions, something that not all teachers do naturally. When we represent these division problems as fractions — 6/2 or 12/4 — the fractions are equivalent; that is, they are two different representations of the same number.

After we had explored this, I wrote on the board: 3 x 2 = 6 and, underneath, 6 x 4 = 24. The teachers noticed that when I doubled both factors, the answer, the product, was four times as large. "Why is this so?" I asked. Again, teachers explored other examples first by themselves and then conferred in groups. In this instance, I wanted teachers to think about multiplication in three ways — as repeated addition (2 + 2 + 2), as the area of a rectangle (with dimensions 3 and 2, the area is 6 square units), and as related to a real-world context (with three pairs of socks, there are six socks in all) — and use these to explain why the product quadruples.

HOW CHILDREN LEARN MATHEMATICS

Along with a deep understanding of the mathematics they have to teach, it's important for teachers to understand how children learn mathematics. There are two distinct aspects of learning mathematics. One aspect of learning calls for making sense of mathematical ideas and skills that are rooted in logic, for which reasoning is both the avenue for building understanding and the source for doing so. The other aspect of learning mathematics calls for learning the terminology and symbols we use to describe mathematical ideas. Terminology and symbols are not rooted in logic but are socially agreed-upon conventions.

Teachers benefit from thinking about ideas we typically think of as elementary mathematics.

These two aspects of learning often merge. For example, the symbols for the numerals 0, 1, 2, 3, and so on up to 9 are agreed-upon conventions. It's important for children to learn to recognize and write these numerals. However, we also want children to connect the numerals to the quantities that they represent, and develop an understanding of each number. They must learn, for example, that 6 is less than 9 but more than 5; that it can be broken apart into two small quantities in various ways, such as 4 and 2 or 1 and 5; that combining 1, 2, and 3 results in a total of 6; and so on. This aspect of learning about numbers is not dependent on social convention but, instead, on logic. It requires children to make sense of how quantities can be taken apart and put together and how they compare to one another. To build this knowledge, children must create their own understanding.

Similarly, learning how and why we use combinations of the 10 numerals to represent other quantities, such as 43, 287, or 1,006, is also based in children's sense making. A child may learn to count quite high and do so with ease, but being able to do so doesn't necessarily indicate that the child understands the place value structure of our number system, has a sense of what the numbers mean, or can perform operations with them.

When teaching a mathematics lesson, teachers must keep in mind whether the lesson is based on a logical structure or on a social convention.

When teaching a mathematics lesson, teachers must keep in mind whether the lesson is based on a logical structure or on a social convention. If it's the former, then it's necessary for children to make sense of ideas for themselves in order to gain understanding. That's because in these situations, the source of the child's understanding is internal. It resides within the child. We can present children with learning opportunities, but then we must rely on the child to construct meaning from the activity, problem, or investigation that we present. This is necessary, for example, for children to understand the numerical concepts mentioned above.

The key is providing children opportunities to build their own understanding.

For example, it's possible to teach children how to "borrow" when subtracting by demonstrating the method and having children practice on numerous examples. While the procedure is a generally agreed-upon convention, it's extremely important for children to learn the logic of the convention and why it works. Without understanding, children make common errors, most typically not regrouping when necessary and merely subtracting smaller from larger numbers, or getting confused when zeroes are involved. And understanding doesn't occur from being told. We must give children ways to make sense of our place value system of numeration and how we use it.

However, if a teacher is teaching a mathematical convention — such as the numeral for the quantity six, the plus sign (+) for adding, or the times sign (x) for multiplying — the source for learning these symbols is external. It exists outside the child. That is, there is no way for a child to figure out or discover the information, because there is no logic or meaning in the symbols themselves. Just as learning any social convention, for example that Thanksgiving always falls on a Thursday, children have to rely on learning terminology and symbols from sources outside themselves — teachers, other adults, classmates, books, TV, and so on.

The implications for applying these two aspects of learning to the instructional choices teachers make are significant. When a lesson, or part of a lesson, addresses a social convention of mathematics, the source of the knowledge is external and it makes sense for teachers to impart the knowledge. In these instances, teaching by telling is appropriate, memorization is often necessary, and practice or reinforcement is usually required.

However, when the ideas being taught are based on logic, the source of the learning is within the child and then teaching by telling is not appropriate. It's actually risky for two reasons. One is that teaching by telling implies that hearing, memorizing, and practicing are the keys for bringing meaning to mathematical ideas and skills. This is not so. The key is providing children opportunities to build their own understanding. Another risk is that this approach to instruction — teaching by telling and asking children to remember, as is necessary for information that has no basis in logic — may give children the message that they aren't supposed to make sense of mathematical ideas or look for the meaning in what they're learning.

Too many students memorize the divide-multiply-subtract-bring-down method for long division of whole numbers, or the invert-and-multiply method for dividing fractions, and do not think, or even imagine, that reasons exist for why the methods work. Learning procedures like these without understanding gives a skewed view of mathematics that can limit a students' immediate and later understanding.

Back to the Coke can. When

teaching about pi, teaching by telling is appropriate for introducing children to the symbol we use to represent this relationship. The source of that information is external to students, so it makes sense to write it on the board and ask children to copy it down and learn it. However, it's not appropriate to teach the formulas in the same way, writing them on the board and asking students to copy them down, memorize them, and practice them on exercises. Students need experiences that help them discover and verify the relationship between the circumference and diameter for many circles and opportunities to apply the relationship to problem-solving situations. Expressing the relationships as the standard formulas should come after understanding is developed.

Extending circles

For more about extending circles in a professional development session or in classroom instruction, see Chapter 7, "Finding the Area of a Circle," in *A Collection of Math Lessons From Grades 6 Through 8*, by Marilyn Burns and Cathy Humphreys (Math Solutions Publications, 1990).

CONNECTING TO CLASSROOM INSTRUCTION

The Coke can exercise is an example of a professional development experience that is useful and effective not only for helping teachers think about the significance of mathematical content and about how children learn, but also for raising issues about effective classroom instruction. What mathematics does the Coke can activity address? Is the activity appropriate for children? For which grade or grades is the experience best suited? What might be the mathematical benefits for children at different grade levels?

The Coke can activity is indeed an appropriate activity for children, and I've used it in many classrooms and at different grade levels. With younger children, I keep the mathematical focus on comparing and measuring, not on the characteristics of circles. For example, 2nd graders gleefully predict as I wrap the yarn around the can and compare its length to the height of different containers. They're then eager to explore other containers on their own. I collect a large supply, labeling them A, B, C, and so on, and cut lengths of yarn for the children to use. Working in pairs, the children measure and compare, record their results, sort the containers into three groups according to whether the distance around each is longer, shorter, or about the same length as the container's height, and then figure out how to represent their results. (Note: Because not all containers are cylindrical, such as plastic cups that are wider at the top than at the bottom, I ask children always to measure around the bottoms.)

For 3rd and 4th graders, I broaden the mathematical focus to include measuring with standard units. I provide them tape measures and rulers and ask them to record the actual measurements of the circumferences and heights of the containers. This not only provides them practice with measuring lengths but also engages them with the important idea of approximation when measuring.

Fifth graders and older students also use standard measures to compare the circumferences of the containers' bottoms with their heights, but I take the activity deeper by also focusing on developing the idea of the constant ratio, pi, of the circumferences and diameters of circles. The students measure the diameters of the bottoms of the containers, divide the measures of their circumferences by their diameters, and then compare the quotients. This introduces them to the idea of pi, the ratio of the circumference and diameter of all circles. Not only are the older students engaged in useful practice with measuring and division, they also gain experience with the approximate nature of measurement while developing understanding of the standard formula for circles, $c = \pi d$.

The Coke can activity can also be a springboard for involving teachers in a more extensive unit on circles that connects concepts from several of the content standards — number, geometry, measurement, and algebra — and engages teachers in firsthand investigations that deepen their mathematical understanding while also modeling for them experiences that are effective and appropriate for classroom instruction. The classroom link is essential to professional development. While we want teachers to learn more mathematics and more about how children learn mathematics, we also want to keep in mind the goal of helping teachers build instructional repertoires that will help them improve their classroom practices.

Second graders gleefully predict as I wrap the yarn around the can and compare its length to the height of different containers.

The Coke can experience is just one small example of a fully realized professional development program for mathematics. However, it exemplifies the characteristics that are appropriate for all professional learning aimed at helping teachers improve their teaching of mathematics. It addresses important mathematical ideas basic to the curriculum, provides opportunities to address how children learn mathematics, and makes clear connections to classroom instruction. These three issues — the mathematics, how children learn, and classroom instructional strategies — are the essential components of effective professional development in mathematics. ■